Infinite Series
(Part-1)

P. Sam Johnson

National Institute of Technology Karnataka (NITK)
Surathkal, Mangalore, India

Infinite Series

An infinite series is the sum of an infinite sequence of numbers

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots .$$

Since there are infinitely many terms to add in an infinite series, we cannot just keep adding to see what comes out. Instead we look at what we get by summing the first n terms of the sequence and stopping.

The sum of the first n terms

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n$$

is an ordinary finite sum and can be calculated by normal addition. It is called the ***n*th partial sum**.

As n gets larger, we expect the partial sums to get closer and closer to a limiting value in the same sense that the terms of a sequence approach a limit.

For example, to assign meaning to an expression like

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots .$$

We add terms one at a time from the beginning and look for a pattern in how these partial sums grow.

Partial sum		Suggestive expression for partial sum	Value
First:	$s_1 = 1$	$2 - 1$	1
Second:	$s_2 = 1 + \frac{1}{2}$	$2 - \frac{1}{2}$	$\frac{3}{2}$
Third:	$s_3 = 1 + \frac{1}{2} + \frac{1}{4}$	$2 - \frac{1}{4}$	$\frac{7}{4}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
nth:	$s_n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}}$	$2 - \frac{1}{2^{n-1}}$	$\frac{2^n - 1}{2^{n-1}}$

Indeed there is a pattern. The partial sums form a sequence whose nth term is

$$s_n = 2 - \frac{1}{2^{n-1}}.$$

This sequence of partial sums converges to 2 because $\lim_{n\to\infty} \frac{1}{2^n} = 0.$

We say "the sum of the infinite series $1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}} + \cdots$ is 2."

Is the sum of any finite number of terms in this series equal to 2? No. Can we actually add an infinite number of terms one by one? No. But we can still define their sum by defining it to be the limit of the sequence of partial sums as $n \to \infty$, in this case 2.

Our knowledge of sequences and limits enables us to break away from the confines of finite sums.

Definition 1 (Infinite Series, *n*th Term, Partial Sum).

Given a sequence of numbers $\{a_n\}$, *an expression of the form*

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is an infinite series. *The number* a_n *is the* nth term *of the series. The sequence* $\{s_n\}$ *defined by*

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ &\vdots \\ s_n &= a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k \end{aligned}$$

is the sequence of partial sums *of the series, the number* s_n *being the* nth partial sum.

Definition 2 (Converges, Sum).

If the sequence of partial sums converges to a limit L, we say that the series converges *and that its* sum *is L.*

In this case, we also write

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n = L.$$

If the sequence of partial sums of the series does not converge, we say that the series diverges.

Notation

When we begin to study a given series

$$a_1 + a_2 + \cdots + a_n + \cdots$$

we might not know whether it converges or diverges.

In either case, it is convenient to use sigma notation to write the series as

$$\sum_{n=1}^{\infty} a_n, \quad \sum_{k=1}^{\infty} a_k, \quad \text{or} \quad \sum a_n.$$

Geometric Series

Geometric series are series of the form

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

in which a and r are fixed real numbers and $a \neq 0$. The series can also be written as $\sum_{n=0}^{\infty} ar^n$. The **ratio** r can be positive, as in

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \left(\frac{1}{2}\right)^{n-1} + \cdots ,$$

or negative, as in

$$1 - \frac{1}{3} + \frac{1}{9} - \cdots + \left(-\frac{1}{3}\right)^{n-1} + \cdots .$$

Case(i) : $r = 1$

If $r = 1$, the nth partial sum of the geometric series is

$$s_n = a + a(1) + a(1)^2 + \cdots + a(1)^{n-1} = na,$$

and the series diverges because $\lim_{n\to\infty} s_n = \pm\infty$ depending on the sign of a.

Case(ii) : $r = -1$

If $r = -1$, the series diverges because the nth partial sums alternate between a and 0.

Case(iii) : $|r| \neq 1$

If $|r| \neq 1$, we can determine the convergence or divergence of the series in the following way.

$$\begin{aligned} s_n &= a + ar + ar^2 + \cdots + ar^{n-1} \\ rs_n &= ar + ar^2 + \cdots + ar^{n-1} + ar^n \\ s_n - rs_n &= a - ar^n \\ s_n(1-r) &= a(1-r^n) \\ s_n &= \frac{a(1-r^n)}{1-r} \quad (r \neq 1). \end{aligned}$$

If $|r| < 1$, then $r^n \to 0$ as $n \to \infty$ and $s_n \to a/(1-r)$. If $|r| > 1$, then $|r^n| \to \infty$ and the series diverges.

Thus the geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ converges if $|r| < 1$ and diverges if $|r| \geq 1$.

Infinite Series

We have determined when a geometric series converges or diverges, and to what value.

The formula

$$\frac{a}{1-r}$$

for the sum of a geometric series applies only when the summation index begins with $n = 1$ in the expression

$$\sum_{n=1}^{\infty} = ar^{n-1}$$

(or with the index $n = 0$ if we write the series as $\sum_{n=0}^{\infty} ar^n$).

Example 3 (Index starts with $n = 1$).

The geometric series with $a = 1/9$ *and* $r = 1/3$ *is*

$$\frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots = \sum_{n=1}^{\infty} \frac{1}{9}\left(\frac{1}{3}\right)^{n-1} = \frac{1/9}{1-(1/3)} = \frac{1}{6}.$$

Example 4 (Index Starts with $n = 0$).

The series

$$\sum_{n=0}^{\infty} \frac{(-1)^n 5}{4^n} = 5 - \frac{5}{4} + \frac{5}{16} - \frac{5}{64} + \cdots$$

is a geometric series with $a = 5$ *and* $r = -1/4$*. It converges to*

$$\frac{a}{1-r} = \frac{5}{1+(1/4)} = 4.$$

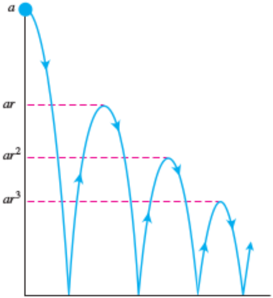
Examples of Geometric Series

Example 5.

1. $$\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} + \cdots$$
   $\longrightarrow$ *Convergent as* $|r| = |1/2| < 1.$
2. $$\sum_{n=0}^{\infty} \frac{(-1)^n}{5^n} = 1 - \frac{1}{5} + \frac{1}{5^2} - \frac{1}{5^3} + \cdots + \frac{(-1)^n}{5^n} + \cdots$$
   $\longrightarrow$ *Convergent as* $|r| = |-1/5| < 1.$
3. $$\sum_{n=0}^{\infty} \frac{(-1)^n 5}{4^n} = 5 - \frac{5}{4} + \frac{5}{4^2} - \frac{5}{4^3} + \cdots + \frac{(-1)^n 5}{4^n} + \cdots$$
   $\longrightarrow$ *Convergent as* $|r| = |-1/4| < 1.$
4. $$\sum_{n=1}^{\infty} \left(\frac{7}{4}\right)^{n-1} = 1 + \frac{7}{4} + \left(\frac{7}{4}\right)^2 + \left(\frac{7}{4}\right)^3 + \cdots + \left(\frac{7}{4}\right)^{n-1} + \cdots$$
   $\longrightarrow$ *Divergent as* $|r| = |7/4| > 1.$

Example 6 (A Bouncing Ball).

We drop a ball from "a" meters above a flat surface. Each time the ball hits the surface after falling a distance h, it rebounds a distance rh, where r is positive but less than 1. Find the total distance the ball travels up and down.



A Bouncing Ball

The total distance is

$$s = a + 2ar + 2ar^2 + 2ar^3 + \cdots = a + \frac{2ar}{1-r} = a\frac{1+r}{1-r}.$$

If $a = 6$ m and $r = 2/3$, for instance, the distance is

$$s = 6\frac{1+(2/3)}{1-(2/3)} = 6\left(\frac{5/3}{1/3}\right) = 30m.$$

Is $0.99999\cdots$ approximately 1, or equal to 1?

Example 7 (Repeating Decimals).

Express the repeating decimal

$$5.232323\cdots$$

as the ratio of two integers.

$$\begin{aligned}
5.232323\cdots &= 5 + \frac{23}{100} + \frac{23}{(100)^2} + \frac{23}{(100)^3} + \cdots \\
&= 5 + \frac{23}{100}\left(\frac{1}{0.99}\right) \\
&= \frac{518}{99}
\end{aligned}$$

Definition 8.

A series of the form $\sum_{n=1}^{\infty}(a_n - a_{n+1})$ *is called a telescoping series.*

Unfortunately, formulas like the one for the sum of a convergent geometric series are rare and we usually have to settle for an estimate of a series' sum. The next example, however, is another case in which we can find the sum exactly.

Example 9 (A Non-geometric Series but Telescoping Series).

Find the sum of the series

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}.$$

Solution

We look for a pattern in the sequence of partial sums that might lead to a formula for s_k. The key observation is the partial fraction decomposition

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1},$$

so

$$\sum_{n=1}^{k} \frac{1}{n(n+1)} = \sum_{n=1}^{k} \left(\frac{1}{n} - \frac{1}{n+1} \right)$$

and

$$s_k = \left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \cdots + \left(\frac{1}{k} - \frac{1}{k+1} \right).$$

Solution (contd...)

Removing parentheses and canceling adjacent terms of opposite sign collapses the sum to

$$s_k = 1 - \frac{1}{k+1}.$$

We now see that $s_k \to 1$ as $k \to \infty$.

The series converges, and its sum is 1:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

Examples of Telescoping Series

Example 10.

1. $$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots + \frac{1}{n(n+1)} + \cdots$$

2. $$\sum_{n=1}^{\infty} \frac{5}{(n+1)(n+2)} = \frac{5}{2.3} + \frac{5}{3.4} + \frac{5}{4.5} + \cdots + \frac{5}{(n+1)(n+2)} + \cdots$$

3. $$\sum_{n=1}^{\infty} \frac{4}{(4n-3)(4n+1)} = \frac{4}{1.5} + \frac{4}{5.9} + \frac{4}{9.13} + \cdots + \frac{4}{(4n-3)(4n+1)} + \cdots$$

4. $$\sum_{n=1}^{\infty} \left[\frac{-1}{\ln(n+1)} + \frac{1}{\ln(n+2)} \right] = \left(-\frac{1}{\ln 2} + \frac{1}{\ln 3} \right) + \left(-\frac{1}{\ln 3} + \frac{1}{\ln 4} \right) + \cdots$$

Example 11.

The sequence of partial sums of the series

$$\sum_{n=1}^{\infty}\left(\frac{1}{n+2}-\frac{1}{n+4}\right)$$

is

$$s_n = \frac{1}{3}+\frac{1}{4}-\frac{1}{n+3}-\frac{1}{n+4}.$$

Since $s_n \to \frac{7}{12}$ *as* $n \to \infty$*, the sum of the series* $\sum_{n=1}^{\infty}\left(\frac{1}{n+2}-\frac{1}{n+4}\right)$ *is* $\frac{7}{12}$.

Necessary condition for convergence

Theorem 12.

If $\sum_{n=1}^{\infty} a_n$ *converges, then* $a_n \to 0$.

Proof :

Let S represent the series' sum and $s_n = a_1 + a_2 + \cdots + a_n$ the nth partial sum.

When n is large, both s_n and s_{n-1} are close to S, so their difference, a_n, is close to zero. More formally,

$$a_n = s_n - s_{n-1} \to S - S = 0.$$

The nth-Term Test for Divergence

The above theorem leads to a test for detecting the kind of divergence, called the nth term test.

Theorem 13 (The nth-Term Test for Divergence).

If $\lim_{n \to \infty} a_n$ *fails to exist or is different from zero, then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Theorem (12) does not say that $\sum_{n=1}^{\infty} a_n$ converges if $a_n \to 0$.

It is possible for a series to diverge when $a_n \to 0$.

One reason that a series may fail to converge is that its terms do not become small.

Example 14 (Partial Sums Outgrow Any Number).

(a) *The series*

$$\sum_{n=1}^{\infty} n^2 = 1 + 4 + 9 + \cdots + n^2 + \cdots$$

diverges because the partial sums grow beyond every number L. After $n = 1$, *the partial sum* $s_n = 1 + 4 + 9 + \cdots + n^2$ *is greater than* n^2.

(b) *The series*

$$\sum_{n=1}^{\infty} \frac{n+1}{n} = \frac{2}{1} + \frac{3}{2} + \frac{4}{3} + \cdots + \frac{n+1}{n} + \cdots$$

diverges because the partial sums eventually outgrow every preassigned number. Each term is greater than 1, so the sum of n terms is greater than n.

Example 15 (Applying the nth-Term Test).

(a) $\displaystyle\sum_{n=1}^{\infty} n^2$ *diverges because* $n^2 \to \infty$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n}$ *diverges because* $\frac{n+1}{n} \to 1$.

(c) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}$ *diverges becaues* $\displaystyle\lim_{n\to\infty} (-1)^{n+1}$ *does not exist.*

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{-n}{2n+5}$ *diverges because* $\displaystyle\lim_{n\to\infty} \frac{-n}{2n+5} = -\frac{1}{2} \neq 0$.

We have seen that if $\sum a_n$ converges, then $a_n \to 0$.

The converse need not be true. The following example illustrates this. $a_n \to 0$ but the series diverges.

Example 16.

The series

$$1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \cdots + \frac{1}{2^n} + \frac{1}{2^n} + \cdots + \frac{1}{2^n} + \cdots$$

diverges because the terms are grouped into clusters that add to 1*, so the partial sums increase without bound.*

However, the terms of the series form a sequence that converges to 0*. We shall see that the harmonic series* $\lim_{n\to\infty} \frac{1}{n}$ *also behaves in this manner.*

Combining Series

Whenever **we have two convergent series**, we can add them term by term, subtract them term by term, or multiply them by constants to make new convergent series.

Theorem 17.

If $\sum a_n = A$ *and* $\sum b_n = B$ *are convergent series, then*

1. *Sum Rule :* $\sum (a_n + b_n) = \sum a_n + \sum b_n = A + B$
2. *Difference Rule :* $\sum (a_n - b_n) = \sum a_n - \sum b_n = A - B$
3. *Constant Multiple Rule :* $\sum ka_n = k \sum a_n = kA$ *(any number k)*

The three rules for series form the analogous rules for sequences.

Proof (Sum Rule)

To prove the Sum Rule for series, let

$$A_n = a_1 + a_2 + \cdots + a_n, \quad B_n = b_1 + b_2 + \cdots + b_n.$$

Then the partial sums of $\sum(a_n + b_n)$ are

$$\begin{aligned} s_n &= (a_1 + a_2) + (a_2 + b_2) + \cdots + (a_n + b_n) \\ &= (a_1 + \cdots + a_n) + (b_1 + \cdots + b_n) \\ &= A_n + B_n. \end{aligned}$$

Since $A_n \to A$ and $B_n \to B$, we have $s_n \to A + B$ by the Sum Rule for sequences.

The proof of the Difference Rule is similar.

To prove the **Constant Multiple Rule** for series, observe that the partial sums of $\sum ka_n$ form the sequence

$$s_n = ka_1 + ka_2 + \cdots + ka_n = k(a_1 + a_2 + \cdots + a_n) = kA_n,$$

which converges to kA by the Constant Multiple Rule for sequences.

As corollaries of the Theorem (17), we have the following results. We omit proof.

Theorem 18.

(a) *Every nonzero constant multiple of a divergent series diverges.*

(b) *If* $\sum a_n$ *converges and* $\sum b_n$ *diverges, then* $\sum(a_n + b_n)$ *and* $\sum(a_n - b_n)$ *both diverge.*

The following example is given to show that $\sum(a_n + b_n)$ can converge when $\sum a_n$ and $\sum b_n$ both diverge.

Example 19.

$\sum a_n = 1 + 1 + 1 + \cdots$ *and* $\sum b_n = (-1) + (-1) + (-1) + \cdots$ *diverge, whereas* $\sum(a_n + b_n) = 0 + 0 + 0 + \cdots$ *converges to 0.*

Properties of convergent series (contd...)

Converse of the sum and difference rules do not hold. That is,

- $\sum(a_n + b_n)$ is convergent $\not\Rightarrow \sum a_n$ and $\sum b_n$ are convergent.
  In other words, $\sum(a_n + b_n)$ may be convergent, but either one or both of $\sum a_n$ and $\sum b_n$ may be divergent.

  Example : Both $\sum a_n = \sum(1)$ and $\sum b_n = \sum(-1)$ are divergent, whereas $\sum(a_n + b_n) = 0$ is convergent.

- $\sum(a_n - b_n)$ is convergent $\not\Rightarrow \sum a_n$ and $\sum b_n$ are convergent.

Is the converse of constant multiple rule true ?

Theorem 20.

If $\sum a_n$ is divergent, then for any constant $k \neq 0$, $\sum (ka_n)$ is also divergent. That is, every non-zero constant multiple of a divergent series is divergent.

Proof.

- Let $\sum a_n$ diverge to $+\infty$ and $\{s_n\}$ be the sequence of partial sums of $\sum (a_n)$. Then $\lim\limits_{n\to\infty} s_n = +\infty$.
- If $\{t_n\}$ is the sequence of partial sums of $\sum (ka_n)$, then
$$\lim_{n\to\infty} t_n = k(\lim_{n\to\infty} s_n) = \begin{cases} +\infty & \text{if } k > 0 \\ -\infty & \text{if } k < 0 \end{cases}$$
Hence, $\sum (ka_n)$ is divergent, where $k \neq 0$.
- Similarly, if $\sum a_n$ diverges to $-\infty$, then $\sum (ka_n)$ is divergent, for $k \neq 0$.

□

Properties of Divergent Series (contd...)

- If $\sum a_n$ converges and $\sum b_n$ diverges, then both $\sum(a_n + b_n)$ and $\sum(a_n - b_n)$ diverge.

- If both $\sum a_n$ and $\sum b_n$ diverge, then $\sum(a_n + b_n)$ and $\sum(a_n - b_n)$ can by of any nature.

- For example, if $\sum a_n = \sum(1)$ and $\sum b_n = \sum(-1)$, then $\sum(a_n + b_n) = \sum(0)$, which is convergent.

  Whereas, if $\sum a_n = \sum(n)$ and $\sum b_n = \sum(n^2)$, then $\sum(a_n + b_n) = \sum(n + n^2)$, which is divergent.

Convergence of Series of Non-negative Terms

If $\sum a_n$ is an infinite series of non-negative terms, that is, $a_n \geq 0$, for all n, then clearly, $s_{n+1} \geq s_n$, for all n and hence $\{s_n\}$ is always monotonically increasing.

Therefore, we have the following by Monotone Convergence Theorem (MCT).

Corollary 21 (To MCT/Non-increasing Sequence Theorem).

If $\sum a_n$ is a series of non-negative terms, then $\sum a_n$ is convergent $\Leftrightarrow$ $\{s_n\}$ is bounded above.

A series of non-negative terms either converges or diverges to $+\infty$.

Example 1 (Harmonic Series).

Discuss the converges of the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots$$

Solution : Let $\sum a_n = \sum \frac{1}{n}$ and $\{s_n\}$ be its sequence of partial sums. Clearly, $a_n \geq 0$, for all n. So, it is enough to verify whether $\{s_n\}$ is bounded above. We have

$$s_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \cdots + \frac{1}{n}.$$

Harmonic Series

Note that

$$
\begin{aligned}
1 + \frac{1}{2} &> \frac{1}{2} \\
\frac{1}{3} + \frac{1}{4} &> \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \\
\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} &> \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2} \\
\vdots \quad & \vdots \quad \vdots \\
\frac{1}{2^n + 1} + \frac{1}{2^n + 2} + \cdots + \frac{1}{2^{n+1}} &> \frac{1}{2^{n+1}} + \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{n+1}} \\
&= \frac{2^n}{2^{n+1} = \frac{1}{2}} \\
\Rightarrow 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^{n+1}} &> \frac{n+1}{2}
\end{aligned}
\tag{1}
$$

In general, if $n = 2^k$, then $s_n > \dfrac{k}{2}$.

$\Rightarrow \{s_n\}$ is not bounded above, hence $\sum a_n$ is not convergent.

Example 22.

(a) *The sum of* $\displaystyle\sum_{n=1}^{\infty} \frac{3^{n-1}-1}{6^{n-1}}$ *is* $\frac{4}{5}$:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{3^{n-1}-1}{6^{n-1}} &= \sum_{n=1}^{\infty}\left(\frac{1}{2^{n-1}}-\frac{1}{6^{n-1}}\right)\\ &= \sum_{n=1}^{\infty}\frac{1}{2^{n-1}}-\sum_{n=1}^{\infty}\frac{1}{6^{n-1}} = 2-\frac{6}{5}=\frac{4}{5}\end{aligned}$$

(b) *The sum of* $\displaystyle\sum_{n=1}^{\infty} \frac{4}{2^n}$ *is* 8 :

$$\sum_{n=1}^{\infty}\frac{4}{2^n} = 4\sum_{n=0}^{\infty}\frac{1}{2^n} = 8$$

Adding or Deleting Terms

We can add a finite number of terms to a series or delete a finite number of terms without altering the series' convergence or divergence, **although in the case of convergence this will usually change the sum.**

If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=k}^{\infty} a_n$ converges for any $k > 1$ and

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_{k-1} + \sum_{n=k}^{\infty} a_n.$$

Conversely, if $\sum_{n=k}^{\infty} a_n$ converges for any $k > 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

Reindexing

As long as we preserve the order of its terms, we can reindex any series without altering its convergence.

To raise the starting value of the index h units, replace the n in the formula for a_n by $n - h$.

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1+h}^{\infty} a_{n-h} = a_1 + a_2 + a_3 + \cdots .$$

Reindexing

To lower the starting value of the index h units, replace the n in the formula for a_n by $n + h$:

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1-h}^{\infty} a_{n+h} = a_1 + a_2 + a_3 + \cdots .$$

It works like a horizontal shift. We saw this in starting a geometric series with the index $n = 0$ instead of the index $n = 1$, but we can use any other starting index value as well.

We usually give preference to indexings that lead to simple expressions.

Example 23 (Reindexing a Geometric Series).

We can write the geometric series

$$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 1 + \frac{1}{2} + \frac{1}{4} + \cdots$$

as

$$\sum_{n=1}^{\infty} \frac{1}{2^{n}} \quad \sum_{n=5}^{\infty} \frac{1}{2^{n-5}}, \quad \text{or even} \quad \sum_{n=-4}^{\infty} \frac{1}{2^{n+4}}.$$

The partial sums remain the same no matter what indexing we choose to use.

Finding nth Partial Sums

Exercise 24.

Find a formula for the nth partial sum of each series and use it to find the series' sum if the series converges.

1. $2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \cdots + \frac{2}{3^{n-1}} + \cdots$
2. $\frac{9}{100} + \frac{9}{100^2} + \frac{9}{100^3} + \cdots + \frac{9}{100^n} + \cdots$
3. $1 - 2 + 4 - 8 + \cdots + (-1)^{n-1} 2^{n-1} + \cdots$
4. $\frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \cdots + \frac{1}{(n+1)(n+2)} + \cdots$

Solution

1. 3
2. $\frac{1}{11}$
3. diverges
4. $\frac{1}{2}$

Write out the first few terms of each series to show how the series starts. Then find the sum of the series.

Exercise 25.

1. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n}$
2. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{5}{2^n} - \frac{1}{3^n} \right)$
3. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{2^n} + \frac{(-1)^n}{5^n} \right)$

Solution

1. $\frac{4}{5}$
2. $\frac{17}{2}$
3. $\frac{17}{6}$

Exercise 26.

Use the nth-Term Test for divergence to show that the series is divergent, or state that the test is inconclusive.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{n(n+1)}{(n+2)(n+3)}$
2. $\displaystyle\sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right)$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{e^n + \pi}$
4. $\displaystyle\sum_{n=1}^{\infty} \ln \frac{1}{n}$
5. $\displaystyle\sum_{n=1}^{\infty} \cos n\pi$

Solution

1. diverges
2. diverges
3. diverges
4. diverges
5. diverges

Exercise 27.

Find a formula for the nth partial sum of the series and use it to determine if the series converges or diverges. If a series converges, find its sum.

1. $\sum_{n=1}^{\infty} \left(\frac{3}{n^2} - \frac{3}{(n+1)^2} \right)$
2. $\sum_{n=1}^{\infty} \left(\ln \sqrt{n+1} - \ln \sqrt{n} \right)$
3. $\sum_{n=1}^{\infty} \left(\tan n - \tan(n-1) \right)$
4. $\sum_{n=1}^{\infty} \left(\cos^{-1}\left(\frac{1}{n+1} \right) - \cos^{-1}\left(\frac{1}{n+2} \right) \right)$
5. $\sum_{n=1}^{\infty} \left(\sqrt{n+4} - \sqrt{n+3} \right)$

Solution

1. converges to 3
2. diverges
3. diverges
4. converges to $-\frac{\pi}{6}$
5. diverges

Exercise 28.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{6}{(2n-1)(2n+1)}$
2. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right)$
3. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{\ln(n+2)} - \frac{1}{\ln(n+1)}\right)$
4. $\displaystyle\sum_{n=1}^{\infty} \left(\tan^{-1}(n) - \tan^{-1}(n+1)\right)$

Solution

1. converges to 3
2. converges to 1
3. converges to $-\frac{1}{\ln 2}$
4. converges to $-\frac{\pi}{4}$

Exercise 29.

Which of the following converge, and which diverge? Give reasons for your answers. If a series converges, find its sum.

1. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3}{2^n}$
2. $\displaystyle\sum_{n=0}^{\infty}\frac{\cos n\pi}{5^n}$
3. $\displaystyle\sum_{n=0}^{\infty}\frac{1}{x^n}, \qquad |x|>1$
4. $\displaystyle\sum_{n=0}^{\infty}\frac{n!}{1000^n}$
5. $\displaystyle\sum_{n=1}^{\infty}\frac{n^n}{n!}$
6. $\displaystyle\sum_{n=1}^{\infty}\ln\left(\frac{n}{2n+1}\right)$
7. $\displaystyle\sum_{n=0}^{\infty}\frac{e^{n\pi}}{\pi^{ne}}$

Solution

1. converges to 1
2. converges to $\frac{5}{6}$
3. converges to $\frac{x}{x-1}$
4. diverges
5. diverges
6. diverges
7. divergent geometric series

Exercise 30.

In each of the following geometric series, write out the first few terms of the series to find a and r, and find the sum of series. Then express the inequality $|r| < 1$ in terms of x and find the values of x for which the inequality holds and the series converges.

1. $\displaystyle\sum_{n=0}^{\infty}(-1)^n x^{2n}$
2. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{2}\left(\frac{1}{3+\sin x}\right)^n$

Solution

1. converges to $\frac{1}{1+x^2}$ for $|x| < 1$
2. converges to $\frac{3+\sin x}{8+2\sin x}$ for all x

Exercise 31.

Find the values of x for which the given geometric series converges. Also, find the sum of the series (as a function of x) for those values of x.

1. $$\sum_{n=0}^{\infty} 2^n x^n$$
2. $$\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n (x-3)^n$$
3. $$\sum_{n=0}^{\infty} \sin^n x$$
4. $$\sum_{n=0}^{\infty} (\ln x)^n$$

Solution

1. converges to $\frac{1}{1-2x}$ for $|x| < \frac{1}{2}$
2. converges to $\frac{2}{x-1}$ for $1 < x < 5$
3. converges to $\frac{1}{1-\sin x}$ for $x \neq (2k+1)\frac{\pi}{2}$, k is an integer
4. converges to $\frac{1}{1-\ln x}$ for $e^{-1} < x < e$

Repeating Decimals

Exercise 32.

Express each of the following numbers as the ratio of two integers.

1. $0.\overline{23} = 0.232323\cdots$
2. $\overline{0.7} = 0.7777\cdots$
3. $3.\overline{142857} = 3.142857142857\cdots$

Solution

1. $\frac{23}{99}$
2. $\frac{7}{9}$
3. $\frac{116{,}402}{37{,}037}$

Exercise 33.

1. *Write the series* $\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)}$ *as a sum beginning with*

   (a) $n = -2$ (b) $n = 0$ (c) $n = 5$

2. *Make up an infinite series of nonzero terms whose sum is*

   (a) 1 (b) -3 (c) 0

3. *Can you make an infinite series of nonzero terms that converges to any number you want? Explain.*
4. *Show by example that* $\sum(a_n/b_n)$ *may diverge even though* $\sum a_n$ *and* $\sum b_n$ *converge and no* b_n *equals 0.*

Solution

1. (a) $$\sum_{n=-2}^{\infty} \frac{1}{(n+4)(n+5)}$$ (b) $$\sum_{n=0}^{\infty} \frac{1}{(n+2)(n+3)}$$ (c) $$\sum_{n=5}^{\infty} \frac{1}{(n-3)(n-2)}$$

2. (a) one example is $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = \frac{\left(\frac{1}{2}\right)}{1-\left(\frac{1}{2}\right)} = 1$
   (b) one example is $-\frac{3}{2} - \frac{3}{4} - \frac{3}{8} - \frac{3}{16} - \cdots = \frac{\left(-\frac{3}{2}\right)}{1-\left(\frac{1}{2}\right)} = -3$
   (c) one example is $1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{8} - \frac{1}{16} - \cdots = 1 - \frac{\left(\frac{1}{2}\right)}{1-\left(\frac{1}{2}\right)} = 0.$
3. The series $\sum_{n=0}^{\infty} k \left(\frac{1}{2}\right)^{n+1}$ is a geometric series whose sum is $\frac{\left(\frac{k}{2}\right)}{1-\left(\frac{1}{2}\right)} = k$ where k can be any positive or negative number.
4. Let $a_n = b_n = \left(\frac{1}{2}\right)^n$. Then $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = 1$, while
   $\sum_{n=1}^{\infty} \left(\frac{a_n}{b_n}\right) = \sum_{n=1}^{\infty} (1)$ diverges.

Exercise 34.

1. *Find convergent geometric series* $A = \sum a_n$ *and* $B = \sum b_n$ *that illustrate the fact that* $\sum a_n b_n$ *may converge without being equal to* AB.
2. *Show by example that* $\sum (a_n/b_n)$ *may converge to something other than* A/B *even when* $A = \sum a_n, B = \sum b_n \neq 0$, *and no* b_n *equals 0.*
3. *If* $\sum a_n$ *converges and* $a_n > 0$ *for all* n, *can anything be said about* $\sum (1/a_n)$? *Give reasons for your answer.*

Solution

1. Let $a_n = b_n = \left(\frac{1}{2}\right)^n$. Then $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = 1$, while $\sum_{n=1}^{\infty} (a_n b_n) = \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n = \frac{1}{3} \neq AB$.
2. Let $a_n = \left(\frac{1}{4}\right)^n$ and $b_n = \left(\frac{1}{2}\right)^n$. Then $A = \sum_{n=1}^{\infty} a_n = \frac{1}{3}$, $B = \sum_{n=1}^{\infty} b_n = 1$ and $\sum_{n=1}^{\infty} \left(\frac{a_n}{b_n}\right) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = 1 \neq \frac{A}{B}$.
3. Yes: $\sum \left(\frac{1}{a_n}\right)$ diverges. The reasoning: $\sum a_n$ converges $\Rightarrow a_n \to 0 \Rightarrow \frac{1}{a_n} \to \infty \Rightarrow \sum \left(\frac{1}{a_n}\right)$ diverges by the nth-Term Test.

Exercise 35.

1. *What happens if you add a finite number of terms to a divergent series or delete a finite number of terms from a divergent series? Give reasons for your answer.*
2. *If $\sum a_n$ converges and $\sum b_n$ diverges, can anything be said about their term-by-term sum $\sum(a_n + b_n)$? Give reasons for your answer.*
3. *Make up a geometric series $\sum ar^{n-1}$ that converges to the number 5*

   (a) $a = 2$ (b) $a = 13/2$

4. *Find the value of b for which $1 + e^b + e^{2b} + e^{3b} + \cdots = 9$.*
5. *For what values of r does the infinite series*

$$1 + 2r + r^2 + 2r^3 + r^4 + 2r^5 + r^6 + \cdots$$

converge? Find the sum of the series when it converges.

Solution

1. Since the sum of a finite number of terms is finite, adding or subtracting a finite number of terms from a series that diverges does not change the divergence of the series.
2. Let $A_n = a_1 + a_2 + \cdots + a_n$ and $\lim_{n\to\infty} A_n = A$. Assume $\sum(a_n + b_n)$ converges to S. Let $S_n = (a_1+b_1)+(a_2+b_2)+\cdots+(a_n+b_n) \Rightarrow S_n = (a_1+a_2+\cdots+a_n)+(b_1+b_2+\cdots+b_n) \Rightarrow b_1 + b_2 + \cdots + b_n = S_n - A_n \Rightarrow \lim_{n\to\infty}(b_1 + b_2 + \cdots + b_n) = S - A \Rightarrow \sum b_n$ converges. This contradicts the assumption that $\sum b_n$ diverges; therefore, $\sum(a_n + b_n)$ diverges.
3. (a) $\frac{2}{1-r} = 5 \Rightarrow \frac{2}{5} = 1 - r \Rightarrow r = \frac{3}{5}$;
   $2 + 2\left(\frac{3}{5}\right) + 2\left(\frac{3}{5}\right)^2 + \cdots = 5.$

   (b) $\frac{\left(\frac{13}{2}\right)}{1-r} = 5 \Rightarrow \frac{13}{10} = 1 - r \Rightarrow r = -\frac{3}{10}$;
   $\frac{13}{2} - \frac{13}{2}\left(\frac{3}{10}\right) + \frac{13}{2}\left(\frac{3}{10}\right)^2 - \frac{13}{2}\left(\frac{3}{10}\right)^3 + \cdots = 5.$
4. $1 + e^b + e^{2b} + \cdots = \frac{1}{1-e^b} = 9 \Rightarrow \frac{1}{9} = 1 - e^b \Rightarrow e^b = \frac{8}{9} \Rightarrow b = \ln\left(\frac{8}{9}\right)$
5. $s_n = 1 + 2r + r^2 + 2r^3 + r^4 + 2r^5 + \cdots + r^{2n} + 2r^{2n+1},\ n = 0, 1, \cdots$
   $\Rightarrow s_n = \left(1 + r^2 + r^4 + \cdots + r^{2n}\right) + \left(2r + 2r^3 + 2r^5 + \cdots + 2r^{2n+1}\right)$
   $\Rightarrow \lim_{n\to\infty} s_n = \frac{1}{1-r^2} + \frac{2r}{1-r^2} = \frac{1+2r}{1-r^2}$, if $|r^2| < 1$ or $|r| < 1$

Exercise 36.

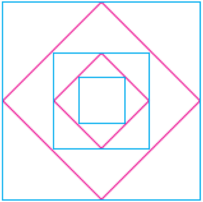
Show that the error $(L - s_n)$ *obtained by replacing a convergent geometric series with one of its partial sums* s_n *is* $ar^n/(1-r)$.

Solution

$L - s_n = \frac{a}{1-r} - \frac{a(1-r^n)}{1-r} = \frac{ar^n}{1-r}$

Exercise 37.

The accompanying figure shows the first five a sequence of squares. The outermost square has an area of $4m^2$. Each of the other squares is obtained by joining the midpoints of the sides of the squares before it. Find the sum of the areas of all the squares.



Solution

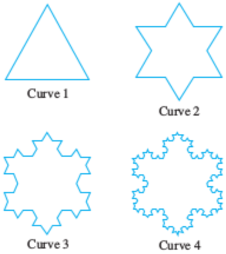
$$\begin{aligned}\text{Area} &= 2^2 + \left(\sqrt{2}\right)^2 + (1)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \cdots \\ &= 4 + 2 + 1 + \frac{1}{2} + \cdots \\ &= \frac{4}{1-\frac{1}{2}} \\ &= 8m^2\end{aligned}$$

Exercise 38 (Helga von Koch's snowflake curve).

Helga von Koch's snow-flake is a curve of infinite length that encloses a region of finite area. To see why this is so, suppose the curve is generated by starting with an equilateral triangle whose sides have length 1.

(a) *Find the length L_n of the nth curve C_n and show that* $\lim_{n\to\infty} L_n = \infty$.

(b) *Find the area A_n of the region enclosed by C_n and calculate* $\lim_{n\to\infty} A_n$.

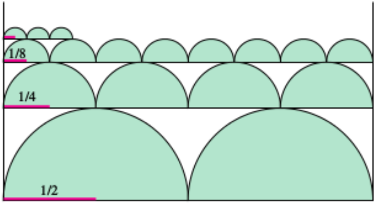


Solution

1. $L_1 = 3, L_2 = 3\left(\frac{4}{3}\right)$,
   $L_3 = 3\left(\frac{4}{3}\right)^2, \ldots, L_n = 3\left(\frac{4}{3}\right)^{n-1} \Rightarrow \lim\limits_{n\to\infty} L_n = \lim\limits_{n\to\infty} 3\left(\frac{4}{3}\right)^{n-1} = \infty$
2. Using the fact that the area of an equilateral triangle of side length s is $\frac{\sqrt{3}}{4}s^2$, we see that $A_1 = \frac{\sqrt{3}}{4}$, $A_2 = A_1 + 3\left(\frac{\sqrt{3}}{4}\right)\left(\frac{1}{3}\right)^2 = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12}$,
   $A_3 = A_2 + 3(4)\left(\frac{\sqrt{3}}{4}\right)\left(\frac{1}{3^2}\right)^2 = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12} + \frac{\sqrt{3}}{27}$,
   $A_4 = A_3 + 3(4)^2\left(\frac{\sqrt{3}}{4}\right)\left(\frac{1}{3^3}\right)^2$, $A_5 = A_4 + 3(4)^3\left(\frac{\sqrt{3}}{4}\right)\left(\frac{1}{3^4}\right)^2, \ldots,$
   $A_n = \frac{\sqrt{3}}{4} + \sum\limits_{k=2}^{n} 3(4)^{k-2}\left(\frac{\sqrt{3}}{4}\right)\left(\frac{1}{3^2}\right)^{k-1} =$
   $\frac{\sqrt{3}}{4} + \sum\limits_{k=2}^{n} 3\sqrt{3}(4)^{k-3}\left(\frac{1}{9}\right)^{k-1} = \frac{\sqrt{3}}{4} + 3\sqrt{3}\left(\sum\limits_{k=2}^{n} \frac{4^{k-3}}{9^{k-1}}\right).$
   $\lim\limits_{n\to\infty} A_n = \lim\limits_{n\to\infty}\left(\frac{\sqrt{3}}{4} + 3\sqrt{3}\left(\sum\limits_{k=2}^{n}\frac{4^{k-3}}{9^{k-1}}\right)\right) = \frac{\sqrt{3}}{4} + 3\sqrt{3}\left(\frac{\frac{1}{36}}{1-\frac{4}{9}}\right) =$
   $\frac{\sqrt{3}}{4} + 3\sqrt{3}\left(\frac{1}{20}\right) = \frac{\sqrt{3}}{4}\left(1 + \frac{3}{5}\right) = \frac{\sqrt{3}}{4}\left(\frac{8}{5}\right) = \frac{8}{5}A_1.$

Exercise 39.

1. *A ball is dropped from a height of* $4m$*. Each time it strikes the pavement after falling from a height of* h *meters it rebounds to a height of* $0.75h$ *meters. Find the total distance the ball travels up and down.*
   Find the total number of seconds the ball is travelling. (Hint: The formula $s = 4.9t^2$ *gives* $t = \sqrt{s/4.9}$*.)*
2. *The accompanying figure shows the first three rows and part of the fourth row of a sequence of rows of semicircles. There are* 2^n *semicircles in the 4th row, each of radius* $1/2''$*. Find the sum of the areas of all the semicircles.*

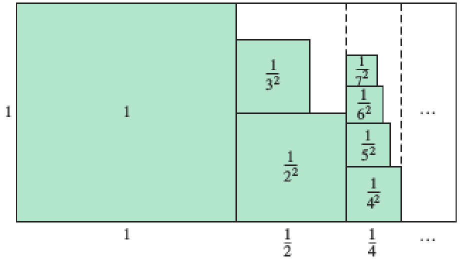


Exercise 40.

The accompanying figure provides an informal proof that

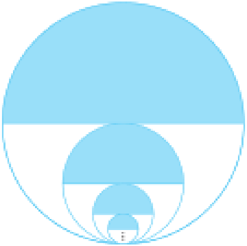
$$\sum_{n=1}^{\infty}(1/n^2)$$

is less than 2*. Explain what is going on.*



Exercise 41.

The largest circle in the accompanying figure has radius 1. Consider the sequence of circles of maximum area inscribed in semicircles of diminishing size. What is the sum of the areas of all of the circles?



Exercise 42 (Drug dosage).

A patient takes a 300mg tablet for the control of high blood pressure every morning at the same time. The concentration of the drug in the patient's system decays exponentially at a constant hourly rate of $k = 0.12$.

1. *How many milligrams of the drug are in the patient's system just before the second tablet is taken? Just before the third tablet is taken?*
2. *In the long run, after taking the medication for at least six months, what quantity of drug is in the patient's body just before taking the next regularly scheduled morning tablet?*

Exercise 43 (The Cantor set).

To construct this set, we begin with the closed interval $[0, 1]$*. From that interval, remove the middle open interval* $(1/3, 2/3)$*, leaving the two closed intervals* $[0, 1/3]$ *and* $[2/3, 1]$*. At the second step we remove the open middle third interval from each of those remaining. From* $[0, 1/3]$ *we remove the open interval* $(1/9, 2/9)$*, and from* $[2/3, 1]$ *we remove* $(7/9, 8/9)$*, leaving behind the four closed intervals* $[0, 1/9], [2/9, 1/3], [2/3, 7/9]$*, and* $[8/9, 1]$*. At the next step, we remove the middle open third interval from each closed interval left behind, so* $(1/27, 2/27)$ *is removed from* $[0, 1/9]$*, leaving the closed intervals* $[0, 1/27]$ *and* $[2/27, 1/9]$*;* $(7/27, 8/27)$ *is removed from* $[2/9, 1/3]$*, leaving behind* $[2/9, 7/27]$ *and* $[8/27, 1/3]$*, and so forth. We continue this process repeatedly without stopping, at each step removing the open third interval from every closed interval remaining behind from the preceding step. The numbers remaining in the interval* $[0, 1]$*, after all open middle third intervals have been removed, are the points in the Cantor set (named after Georg Cantor, 1845-1918). The set has some interesting properties.*

1. *The Cantor set contains infinitely many numbers in* $[0, 1]$*. List* 12 *numbers that belong to the Cantor set.*
2. *Show, by summing an appropriate geometric series, that the total length of all the open middle third intervals that have been removed from* $[0, 1]$ *is equal to* 1*.*

Integral Test

Integral Test

Given a series $\sum a_n$, we have two questions:

1. Does the series converge?
2. If it converges, what is its sum?

We answer these questions by making a connection to the convergence of the improper integral

$$\int_1^\infty f(x)\ dx.$$

However, as a practical matter the second question is also important, and we will discuss it now.

Integral Test

We study series that do not have negative terms. The reason for this restriction is that the partial sums of these series form nondecreasing sequences, and nondecreasing sequences that are bounded from above always converge. To show that a series of nonnegative terms converges, we need only show that its partial sums are bounded from above.

It may at first seem to be a drawback that this approach establishes the fact of convergence without producing the sum of the series in question. Surely it would be better to compute sums of series directly from formulas for their partial sums. But in most cases such formulas are not available, and in their absence we have to turn instead to the two step procedure of first establishing convergence and then approximating the sum.

Nondecreasing Partial Sums

Suppose that

$$\sum_{n=1}^{\infty} a_n$$

is an infinite series with

$$a_n \geq 0 \quad \text{for all } n.$$

Then each partial sum is greater than or equal to its predecessor because $s_{n+1} = s_n + a_n$:

$$s_1 \leq s_2 \leq s_3 \leq \cdots \leq s_n \leq s_{n+1} \leq \cdots .$$

Since the partial sums form a nondecreasing sequence, the Nondecreasing Sequence Theorem tells us the series will converge if and only if the partial sums are bounded from above.

Corollary of Nondecreasing Sequence Theorem

Corollary 44.

A series $\sum_{n=1}^{\infty} a_n$ *of nonnegative terms converges if and only if its partial sums are bounded from above.*

Example 45 (The Harmonic Series).

The series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

is called the harmonic series.

The harmonic series is divergent, but this does not follow from the nth-Term Test.

Harmonic Series

The nth term $\frac{1}{n}$ does go to zero, but the series still diverges.

The reason it diverges is because there is no upper bound for its partial sums.

To see why, group the terms of the series in the following way:

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{16}\right) + \cdots$$

Harmonic Series

The sum of the **first two terms** $3/2$.

The sum of the **next two terms** is $1/3 + 1/4$, which is greater than $1/4 + 1/4 = 1/2$.

The sum of the **next four terms** is $1/5 + 1/6 + 1/7 + 1/8$, which is greater than $1/8 + 1/8 + 1/8 + 1/8 = 1/2$.

The sum of the **next eight terms** is $1/9 + 1/10 + 1/11 + 1/12 + 1/13 + 1/14 + 1/15 + 1/16$, which is greater than $8/16 = 1/2$.

The sum of the **next** 16 **terms** is greater than $16/32 = 1/2$, and so on.

Harmonic Series

In general, the sum of 2^n terms ending with $1/2^{n+1}$ is greater than $2^n/2^{n+1} = 1/2$.

The sequence of partial sums is not bounded from above: If $n = 2^k$, the partial sum s_n is greater than $k/2$.

Therefore the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ **diverges.**

Note that the sequence $\left\{\frac{1}{n}\right\}$ converges.

Don't get confused with the terms "sequence" and "series".

Integral Test

We introduce the Integral Test with a series that is related to the harmonic series, but whose nth term is $1/n^2$ instead of $1/n$.

Example 46 (Does the following series converge?).

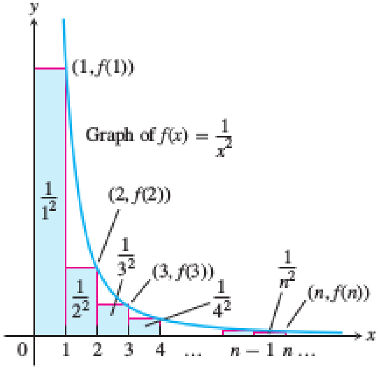
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2} + \cdots .$$

Solution :

We determine the convergence of $\sum_{n=1}^{\infty} \frac{1}{n^2}$ by comparing it with

$$\int_1^{\infty} \frac{1}{x^2}\ dx.$$

To carry out the comparison, we think of the terms of the series as values of the function $f(x) = 1/x^2$ and interpret these values as the areas of rectangles under the curve $y = 1/x^2$.



The sum of the areas of the rectangles under the graph of $f(x) = 1/x^2$ is less than the area under the graph.

Integral Test

From the figure, we have

$$\begin{aligned} s_n &= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \\ &= f(1) + f(2) + f(3) + \cdots + f(n) \\ &< f(1) + \int_1^n \frac{1}{x^2}\, dx \\ &< 1 + \int_1^\infty \frac{1}{x^2}\, dx \\ &< 1 + 1 = 2. \end{aligned}$$

Thus the partial sums of $\sum_{n=1}^{\infty} \frac{1}{n^2}$ are bounded from above (by 2) and the series converges. The sum of the series is known to be $\pi^2/6 \approx 1.64493$.

Integral Test

Theorem 47 (Integral Test).

Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous, positive, decreasing function of x for all $x \geq N$, where N is a positive integer. Then the series

$$\sum_{n=N}^{\infty} a_n$$

and the integral

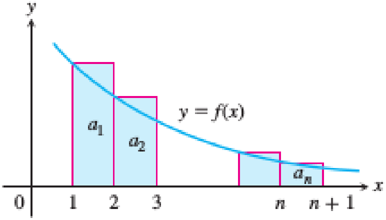
$$\int_N^{\infty} f(x)\, dx$$

both converge or both diverge.

Proof of Integral Test

We establish the test for the case $N = 1$. The proof for general N is similar. We start with the assumption that f is a decreasing function with $f(n) = a_n$ for every n.

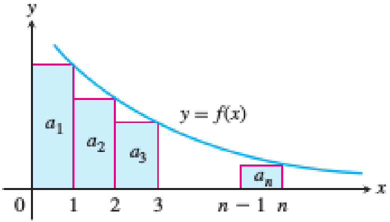
This leads us to observe that the rectangles in the figure, which have areas $a_1, a_2, \ldots, a_n$, collectively enclose more area than that under the curve $y = f(x)$ from $x = 1$ to $x = n + 1$.



That is,

$$\int_1^{n+1} f(x)\, dx \leq a_1 + a_2 + \cdots + a_n.$$

In the figure, the rectangles have been faced to the left instead of to the right.



If we do not consider the first rectangle, of area a_1, we see that

$$a_2 + a_3 + \cdots + a_n \leq \int_1^n f(x)\, dx.$$

If we include a_1, we have

$$a_1 + a_2 + \cdots + a_n \leq a_1 + \int_1^n f(x)\, dx.$$

Proof of the Integral Test (contd . . .)

Combining these results gives

$$\int_1^{n+1} f(x)\, dx \leq a_1 + a_2 + \cdots + a_n \leq a_1 + \int_1^n f(x)\, dx.$$

These inequalities hold for each n, and continue to hold as $n \to \infty$.

If $\int_1^\infty f(x)\, dx$ is finite, the right-hand inequality shows that $\sum a_n$ is finite.

If $\int_1^\infty f(x)\, dx$ is infinite, the left-hand inequality shows that $\sum a_n$ is infinite.

Hence the series and the integral are both finite or both infinite.

Example 48 (The p-series).

Show that the p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

(p is a real constant) converges if $p > 1$, and diverges if $p \leq 1$.

The p-series : Solution :

If $p > 1$, then $f(x) = 1/x^p$ is a positive decreasing function of x. Since

$$\begin{aligned}
\int_1^\infty \frac{1}{x^p}\,dx &= \int_1^\infty x^{-p}\,dx = \lim_{b\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_1^b \\
&= \frac{1}{1-p}\lim_{b\to\infty}\left(\frac{1}{b^{p-1}} - 1\right) \\
&= \frac{1}{1-p}(0-1) = \frac{1}{p-1},
\end{aligned}$$

the series converges by the Integral Test.

We emphasize that the sum of the p-series is not $1/(p-1)$. The series converges, but we do not know the value it converges to.

If $p < 1$, then $1 - p > 0$ and

$$\int_1^{\infty} \frac{1}{x^p}\, dx = \frac{1}{1-p} \lim_{b\to\infty} (b^{1-p} - 1) = \infty.$$

The series diverges by the Integral Test.

If $p = 1$, we have the (divergent) harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots .$$

We have convergence for $p > 1$ but divergence for every other value of p.

The p-Series Test

The p-series with $p = 1$ is the **harmonic series**.

The p-Series Test shows that the harmonic series is just *barely* divergent; if we increase p to 1.000000001, for instance, the series converges!

The slowness with which the partial sums of the harmonic series approaches infinity is impressive.

For instance, it takes about $178,482,301$ terms of the harmonic series to move the partial sums beyond 20. It would take your calculator several weeks to compute a sum with this many terms.

A Convergent Series

Example 49.

The series

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

converges by the Integral Test. The function $f(x) = 1/(x^2+1)$ *is positive, continuous, and decreasing for* $x \geq 1$, *and*

$$\begin{aligned} \int_1^{\infty} \frac{1}{x^2+1}\, dx &= \lim_{b\to\infty} [\arctan x]_1^b = \lim_{b\to\infty} [\arctan b - \arctan 1] \\ &= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}. \end{aligned}$$

Again we emphasize that $\pi/4$ *is not the sum of the series. The series converges, but we do not know the value of its sum.*

Determining Convergence or Divergence

Example 50.

Determine the convergence or divergence of the series.

(a) $\sum_{n=1}^{\infty} ne^{-n^2}$ (b) $\sum_{n=1}^{\infty} \frac{1}{2^{\ln n}}$

Solution :

(a) We apply the Integral Test and find that

$$\int_1^{\infty} \frac{x}{e^{x^2}} dx = \frac{1}{2} \int_1^{\infty} \frac{du}{e^u} \qquad u = x^2, du = 2xdx$$

$$= \lim_{b \to \infty} \left[-\frac{1}{2} e^{-u} \right]_1^b = \lim_{b \to \infty} \left(-\frac{1}{2e^b} + \frac{1}{2e} \right) = \frac{1}{2e}.$$

Since the integral converges, the series also converges.

(b) Again applying the Integral Test,

$$\int_1^{\infty} \frac{dx}{2^{\ln x}} = \int_0^{\infty} \frac{e^u du}{2^u} \quad u = \ln x, x = e^u, dx$$

$$= e^u du = \int_0^{\infty} \left(\frac{e}{2} \right)^u du = \lim_{b \to \infty} \frac{1}{\ln \left(\frac{e}{2} \right)} \left(\left(\frac{e}{2} \right)^b - 1 \right) = \infty. \quad (e/2) > 1$$

The improper integral diverges, so the series diverges also.

Error Estimation

For some convergent series, such as the geometric series or the telescoping series, we can actually find the total sum of the series. That is, we can find the limiting value S of the sequence of partial sums.

For most convergent series, however, we cannot easily find the total sum. Nevertheless, we can *estimate* the sum by adding the first n terms to get s_n, but we need to know how far off s_n is from the total sum S.

An approximation to a function or to a number is more useful when it is accompanied by a bound on the size of the worst possible error that could occur. With such an error bound we can try to make an estimate or approximation that is close enough for the problem at hand.

Without a bound on the error size, we are just guessing and hoping that we are close to the actual answer. We now show a way to bound the error size using integrals.

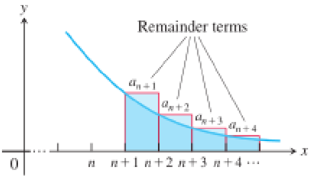
Error Estimation

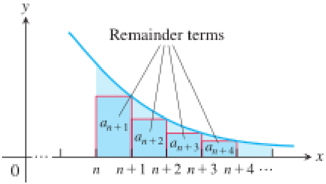
Suppose that a series $\sum a_n$ with positive terms is shown to be convergent by the Integral Test, and we want to estimate the size of the **remainder** R_n measuring the difference between the total sum S of the series and its nth partial sum s_n. That is, we wish to estimate

$$R_n = S - s_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots .$$

To get a lower bound for the remainder, we compare the sum of the areas of the rectangles with the area under the curve $y = f(x)$ for $x \geq n$. We see that

$$R_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots \geq \int_{n+1}^{\infty} f(x)dx.$$





Similarly, from the above figure, we find an upper bound with

$$R_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots \leq \int_n^{\infty} f(x)dx.$$

These comparisons prove the following result, giving bounds on the size of the remainder. The remainder when using n terms is

(a) larger than the integral f over $[n+1, \infty)$.

(b) smaller than the integral of f over $[n, \infty)$.

Bounds for the Remainder in the Integral Test

Suppose $\{a_k\}$ is a sequence of positive terms with $a_k = f(k)$, where f is a continuous positive decreasing function of x for all $x \geq n$, and that $\sum a_n$ converges to S. Then the remainder $R_n = S - s_n$ satisfies the inequalities

$$\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_{n}^{\infty} f(x)dx. \tag{2}$$

If we add the partial sum s_n to each side of the inequalities in (4), we get

$$s_n + \int_{n+1}^{\infty} f(x)dx \leq S \leq s_n + \int_{n}^{\infty} f(x)dx \tag{3}$$

since $s_n + R_n = S$. The inequalities in (3) are useful for estimating the error in approximating the sum of a series known to converge by the Integral Test.

Bounds for the Remainder in the Integral Test

The error can be no larger than the length of the interval containing S, with endpoints given by (3).

Example 51.

Estimate the sum of the series $\sum(1/n^2)$ *using the inequalities in* (3) *and* $n = 10$.

Solution : We have that

$$\int_n^\infty \frac{1}{x^2}dx = \lim_{b\to\infty}\left[-\frac{1}{x}\right]_n^b = \lim_{b\to\infty}\left(-\frac{1}{b}+\frac{1}{n}\right) = \frac{1}{n}.$$

Using this result with the inequalities in (3), we get

$$s_{10} + \frac{1}{11} \leq S \leq s_{10} + \frac{1}{10}.$$

Solution (contd...)

Taking $s_{10} = 1 + (1/4) + (1/9) + (1/16) + \cdots + (1/100) \approx 1.54977$, these last inequalities give

$$1.64068 \leq S \leq 1.64977.$$

If we approximate the sum S by the midpoint of this interval, we find that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \approx 1.6452.$$

The error in this approximation is then less than half the length of the interval, so the error is less than 0.005.

Using a trigonometric *Fourier series* (studied in advanced calculus), it can be shown that S is equal to $\pi^2/6 \approx 1.64493$.

Exercise 52 (Applying the Integral Test).

Use the Integral Test to determine if the following series converge or diverge. Be sure to check that the conditions of the Integral Test are satisfied.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{0.2}}$
2. $\displaystyle\sum_{n=1}^{\infty} e^{-2n}$
3. $\displaystyle\sum_{n=2}^{\infty} \frac{\ln(n^2)}{n}$
4. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{e^{n/3}}$
5. $\displaystyle\sum_{n=2}^{\infty} \frac{n-4}{n^2-2n+1}$

Solution

1. $f(x) = \frac{1}{x^{0.2}}$ is positive, continuous and decreasing for $x \geq 1$. By the Integral Test, the given series diverges.
2. The function is decreasing for $x \geq 1$. By the Integral Test, the given series converges.
3. The function is decreasing for $x \geq 3$. By the Integral Test, the given series diverges.
4. The function is decreasing for $x \geq 7$. By the Integral Test, the given series converges.
5. The function is decreasing for $x \geq 8$. By the Integral Test, the given series diverges.

Exercise 53 (Determining Convergence or Divergence).

Which of the series converge, and which diverge? Give reasons for your answers. (When you check an answer, remember that there may be more than one way to determine the series' convergence or divergence.)

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{10^n}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1}$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{-2}{n\sqrt{n}}$
4. $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{\sqrt{n}}$
5. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n+1}$
6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}(\sqrt{n}+1)}$

Solution

1. converges
2. diverges
3. converges
4. diverges
5. diverges
6. diverges

Exercise 54 (Determining Convergence or Divergence).

Which of the series converge, and which diverge? Give reasons for your answers. (When you check an answer, remember that there may be more than one way to determine the series' convergence or divergence.)

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(\ln 3)^n}$
2. $\displaystyle\sum_{n=3}^{\infty} \frac{(1/n)}{(\ln n)\sqrt{\ln^2 n - 1}}$
3. $\displaystyle\sum_{n=1}^{\infty} n \tan \frac{1}{n}$
4. $\displaystyle\sum_{n=1}^{\infty} \frac{8 \tan^{-1} n}{1+n^2}$
5. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

Solution

1. diverges
2. converges
3. diverges
4. converges
5. converges

Exercise 55.

1. *For what values of a, if any, does the series* $\displaystyle\sum_{n=1}^{\infty}\left(\frac{a}{n+2}-\frac{1}{n+4}\right)$ *converge?*
2. *Are there any values of x for which* $\displaystyle\sum_{n=1}^{\infty}(1/(nx))$ *converges? Give reasons for your answer.*

Solution

1. $\int_1^{\infty}\left(\frac{a}{x+2}-\frac{1}{x+4}\right)dx = \lim_{b\to\infty}\left[a\ln|x+2|-\ln|x+4|\right]_1^b =$
   $\lim_{b\to\infty}\ln\frac{(b+2)^a}{b+4}-\ln\left(\frac{3^a}{5}\right)$;

   But $\lim_{b\to\infty}\frac{(b+2)^a}{b+4} = a\lim_{b\to\infty}(b+2)^{a-1} = \begin{cases}\infty, & a>1\\ 1, & a=1\end{cases}$

   Hence the series converges if $a=1$ (but do not conclude that it converges to $\ln(\frac{5}{3})$) and diverges to ∞ if $a>1$. If $a<1$, the terms of the series eventually become negative and the Integral Test does not apply. From that point on, however, the series behaves like a negative multiple of the harmonic series, and so it diverges.

2. No, because $\sum_{n=1}^{\infty}\frac{1}{nx} = \frac{1}{x}\sum_{n=1}^{\infty}\frac{1}{n}$ and $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.

Exercise 56.

Is it true that if $\sum_{n=1}^{\infty} a_n$ *is a divergent series of positive numbers then there is also a divergent series*

$$\sum_{n=1}^{\infty} b_n$$

of positive numbers with $b_n < a_n$ *for every* n*?*

(a) *Is there a "smallest" divergent series of positive numbers? Give reasons for your answers.*

(b) *Is there a "largest" convergent series of positive numbers? Explain.*

Solution

(a) No. If $\sum_{n=1}^{\infty} a_n$ is a divergent series of positive numbers, then $\left(\frac{1}{2}\right) \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \left(\frac{a_n}{2}\right)$ also diverges and $\frac{a_n}{2} < a_n$. There is no smallest divergent series of positive numbers: for any divergent series $\sum_{n=1}^{\infty} a_n$ of positive numbers $\sum_{n=1}^{\infty} \left(\frac{a_n}{2}\right)$ has smaller terms and still diverges.

(b) No, if $\sum_{n=1}^{\infty} a_n$ is a convergent series of positive numbers, then $2\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} 2a_n$ also converges, and $2a_n \geq a_n$. There is no largest convergent series of positive numbers.

Exercise 57 (The Cauchy Condensation Test).

Let $\{a_n\}$ be a non-increasing sequence ($a_n \geq a_{n+1}$ for all n) of positive terms that converges to 0. Then prove that $\sum a_n$ converges if and only if $\sum 2^n a_{2^n}$ converges.

Proof of Cauchy Condensation Test

Let $A_n = \sum_{k=1}^{n} a_k$ and $B_n = \sum_{k=1}^{n} 2^k a_{2^k}$ where $\{a_k\}$ is a non-increasing sequence of positive terms converging to 0.

Note that $\{A_n\}$ and $\{B_n\}$ are non-decreasing sequences of positive terms. Now

$$\begin{aligned}
B_n &= 2a_2 + 4a_4 + 8a_8 + \cdots + 2^n a_{2^n} \\
&= 2a_2 + 2a_4 + 2a_4 + (2a_8 + 2a_8 + 2a_8 + 2a_8) + \cdots + (2^n a_{2^n} + \cdots + 2^{n-1} \text{ (terms)}) \\
&\leq 2a_1 + 2a_2 + (2a_3 + 2a_4) + (2a_5 + 2a_6 + 2a_7 + 2a_8) + \cdots \\
&\qquad \cdots + 2a_{2^{n-1}} + 2a_{2^{n-1}+1} + \cdots + 2^n a_{2^n} \\
&= 2A_{2^n} \\
&\leq 2\sum_{k=1}^{\infty} a_k
\end{aligned}$$

Therefore if $\sum a_k$ converges, then $\{B_n\}$ is bounded above.

Thus $\sum 2^k a_{2^k}$ converges.

Proof of Cauchy Condensation Test (contd...)

Conversely,

$$\begin{aligned} A_n &= a_1 + (a_2 + a_3) + (a_4 + \cdots + a_7) + \cdots + a_n \\ &\leq a_1 + (a_2 + a_3) + (a_4 + \cdots + a_7) + \cdots + a_{2^{n+1}} \\ &= a_1 + (a_2 + a_3) + (a_4 + \cdots + a_7) + \cdots + (a_{2^n} + \cdots + a_{2^{n+1}}) \\ &< a_1 + 2a_2 + 4a_4 + \cdots + 2^n a_{2^n} \\ &= a_1 + B_n \leq a_1 + \sum_{k=1}^{\infty} 2^k a_{2^k}. \end{aligned}$$

Therefore, if $\sum_{k=1}^{\infty} 2^k a_{2^k}$ converges, then $\{A_n\}$ is bounded above and hence converges.

Example 58.

$\sum(1/n)$ *diverges because* $\sum 2^n \cdot (1/2^n) = \sum 1$ *diverges.*

Exercise 59.

1. *Use the Cauchy condensation test to show that*
   (a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ *diverges;*
   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ *converges if* $p > 1$ *and diverges if* $p \leq 1$.

Solution

(a) $a_{(2^n)} = \frac{1}{2^n \ln(2^n)} = \frac{1}{2^n \cdot n(\ln 2)} \Rightarrow \sum_{n=2}^{\infty} 2^n a_{2^n} = \sum_{n=2}^{\infty} 2^n \frac{1}{2^n \cdot n(\ln 2)} = \frac{1}{\ln 2} \sum_{n=2}^{\infty} \frac{1}{n}$, which diverges $\Rightarrow \sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges.

(b) $a_{(2^n)} = \frac{1}{2^{np}} \Rightarrow \sum_{n=1}^{\infty} 2^n a_{2^n} = \sum_{n=1}^{\infty} 2^n \cdot \frac{1}{2^{np}} = \sum_{n=1}^{\infty} \frac{1}{(2^n)^{p-1}} = \sum_{n=1}^{\infty} \left(\frac{1}{2^{p-1}}\right)^n$, a geometric series that converges if $\frac{1}{2^{p-1}} < 1$ or $p > 1$, but diverges if $p \leq 1$.

Exercise 60.

1. *Show that*

$$\int_2^\infty \frac{dx}{x(\ln x)^p}$$

(p is a positive constant) converges if and only if $p > 1$.

2. *What implications does the fact in the above exercise have for the convergence of the series*

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}?$$

Give reasons for your answer.

Solution

1. $\int_2^\infty \frac{dx}{x(\ln x)^p}$; $\begin{bmatrix} u = \ln x \\ du = \frac{dx}{x} \end{bmatrix} \to \int_{\ln 2}^\infty u^{-p} du = \lim_{b\to\infty} \left[\frac{u^{-p+1}}{-p+1}\right]_{\ln 2}^b =$ $\lim_{b\to\infty} \left(\frac{1}{1-p}\right) [b^{-p+1} - (\ln 2)^{-p+1}] = \begin{cases} \frac{1}{p-1}(\ln 2)^{-p+1}, & p > 1 \\ \infty, & p < 1 \end{cases} \Rightarrow$ the improper integral converges if $p > 1$ and diverges if $p < 1$.
   For $p = 1$: $\int_2^\infty \frac{dx}{x \ln x} = \lim_{b\to\infty} [\ln(\ln x)]_2^b = \lim_{b\to\infty} [\ln(\ln b) - \ln(\ln 2)] = \infty$, so the improper integral diverges if $p = 1$.
2. Since the series and the integral converge or diverge together, $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges if and only if $p > 1$.

Theorem 61 (Logarithmic p-series Test).

Let p be a positive constant. The series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$$

converges if and only if $p > 1$.

Exercise 62 (Logarithmic p-series).

Use Logarithmic p-series test determine which of the following series converge and which diverge. Support your answer in each case.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{1.01}}$

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln(n^3)}$

Solution

(a) $p = 1.01 \Rightarrow$ the series converges

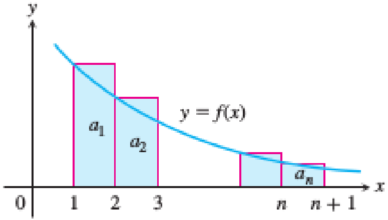
(b) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n^3)} = \frac{1}{3} \sum_{n=2}^{\infty} \frac{1}{n(\ln n)}$; $p = 1 \Rightarrow$ the series diverges

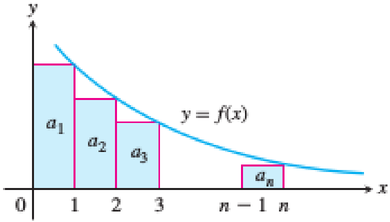
The figures suggest that as n increases there is little change in the difference between the sum

$$1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

and the integral

$$\ln n = \int_1^n \frac{1}{x}\,dx.$$





Euler's Constant

By taking $f(x) = 1/x$ in the proof of "The Integral Test", show that

$$\ln(n+1) \leq 1 + \frac{1}{2} + \cdots + \frac{1}{n} \leq 1 + \ln n$$

or

$$0 < \ln(n+1) - \ln n \leq 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n \leq 1.$$

Thus, the sequence

$$a_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n$$

is bounded from below and from above.

Exercise 63 (Euler's Constant).

Show that

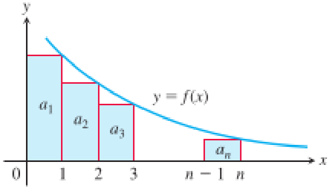
$$\frac{1}{n+1} < \int_n^{n+1} \frac{1}{x}\, dx = \ln(n+1) - \ln n.$$

and use this result to show that the sequence $\{a_n\}$ *defined by* $a_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n$ *is decreasing. Since a decreasing sequence that is bounded from below converges, the numbers* a_n *converge:*

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n \to \gamma.$$

The number γ*, whose value is* $0.5772\cdots$ *, is called* Euler's constant. *In contrast to other special numbers like* π *and* e*, no other expression with a simple law of formulation has ever been found for* γ*.*

Solution



From the above graph with $f(x) = \frac{1}{x}$,
$\frac{1}{n+1} < \int_n^{n+1} \frac{1}{x} dx = \ln(n+1) - \ln n \Rightarrow 0 > \frac{1}{n+1} - [\ln(n+1) - \ln n] = \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1} - \ln(n+1)\right) - \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n\right)$. If we define $a_n = 1 + \frac{1}{2} = \frac{1}{3} + \frac{1}{n} - \ln n$, then
$0 > a_{n+1} - a_n \Rightarrow a_{n+1} < a_n \Rightarrow \{a_n\}$ is a decreasing sequence of nonnegative terms.

Exercise 64.

Use the integral test to show that

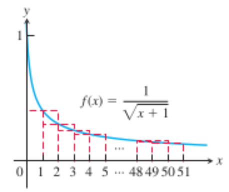
$$\sum_{n=0}^{\infty} e^{-n^2}$$

converges.

Solution

$e^{-x^2} \leq e^{-x}$ for $x \geq 1$, and
$\int_1^\infty e^{-x} dx = \lim_{b \to \infty} [-e^{-x}]_1^b = \lim_{b \to \infty} (-e^{-b} + e^{-1}) = e^{-1} \Rightarrow \int_1^\infty e^{-x^2} dx$
converges by the Comparison Test for improper integrals
$\Rightarrow \sum_{n=0}^{\infty} e^{-n^2} = 1 + \sum_{n=1}^{\infty} e^{-n^2}$ converges by the Integral Test.

Exercise 65.



1. *Use the accompanying graph to show that the partial sum* $s_{50} = \sum_{n=1}^{50}(1/\sqrt{n+1})$ *satisfies*

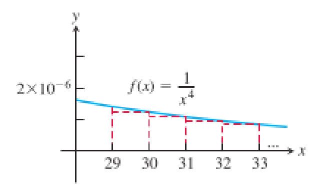
$$\int_1^{51} \frac{1}{\sqrt{x+1}}dx < s_{50} < \int_0^{50} \frac{1}{\sqrt{x+1}}dx.$$

   Conclude that $11.5 < s_{50} < 12.3$.

2. *What should* n *be in order that the partial sum*

$$s_n = \sum_{i=1}^{n}(1/\sqrt{i+1}) \text{ satisfy } s_n > 1000?$$

Exercise 66.



1. *Use the accompanying graph to find an upper bound for the error if* $s_{30} = \sum_{n=1}^{30}(1/n^4)$ *is used to estimate the value of* $\sum_{n=1}^{\infty}(1/n^4)$.
2. *Find* n *so that the partial sum* $s_n = \sum_{i=1}^{n}(1/i^4)$ *estimates the value of* $\sum_{n=1}^{\infty}(1/n^4)$ *with an error of at most* 0.000001.

Exercise 67.

1. *Estimate the value of* $\sum_{n=1}^{\infty}(1/n^3)$ *to within* 0.01 *of its exact value.*
2. *Estimate the value of* $\sum_{n=2}^{\infty}\left(1/(n^2+4)\right)$ *to within* 0.1 *of its exact value.*
3. *How many terms of the convergent series* $\sum_{n=1}^{\infty}(1/n^{1.1})$ *should be used to estimate its value with error at most* 0.00001*?*
4. *How many terms of the convergent series* $\sum_{n=4}^{\infty}1/(n(\ln n)^3)$ *should be used to estimate its value with error at most* 0.01*?*

Exercise 68.

1. (a) *For the series* $\sum(1/n^3)$, *use the inequalities in Equation (3) with* $n = 10$ *to find an interval containing the sum* S.
   (b) *Use the midpoint of the interval found in part (a) to approximate the sum of the series. What is the maximum error for your approximation?*
2. *Repeat the above exercise using the series* $\sum(1/n^4)$.
3. Area *Consider the sequence* $\{1/n\}_{n=1}^{\infty}$. *On each subinterval* $(1/(n+1), 1/n)$ *within the interval* $[0, 1]$, *erect the rectangle with area* a_n *having height* $1/n$ *and width equal to the length of the subinterval. Find the total area* $\sum a_n$ *of all the rectangles.*
4. Area *Repeat the above exercise, using trapezoids instead of rectangles. That is, on the subinterval* $(1/(n+1), 1/n)$, *let* a_n *denote the area of the trapezoid having heights* $y = 1/(n+1)$ *at* $x = 1/(n+1)$ *and* $y = 1/n$ *at* $x = 1/n$.

Comparison Tests

Direct Comparison Test

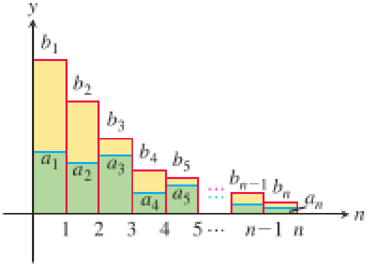
We shall now discuss how to determine the convergence of series by comparing their terms to those of a series whose convergence is known.

We shall also discuss a comparison test (called, **limit comparison test**) that is particularly useful for series in which nth term of the series is a rational function on n.

Theorem 69 (Direct Comparison Test).

Let $\sum a_n$ and $\sum b_n$ be two series with $0 \leq a_n \leq b_n$ for all $n \geq N$.

1. *If $\sum b_n$ converges, then $\sum a_n$ also converges.*
2. *If $\sum a_n$ diverges, then $\sum b_n$ also diverges.*



Proof of Part 1

Consider, for $n \geq N$,

$$\begin{aligned} A_n &= a_1 + a_2 + \cdots + a_{N-1} + a_N + a_{N+1} + \cdots + a_n \\ &\leq (a_1 + a_2 + \cdots + a_{N-1}) + (b_N + b_{N+1} + \cdots + b_n) \\ &= \sum_{n=1}^{N-1} a_n + \sum_{n=N}^{\infty} b_n \leq \sum_{n=1}^{N-1} a_n + \sum_{n=1}^{\infty} b_n \\ &= \sum_{n=1}^{N-1} a_n + B, \qquad \left(\text{since } \sum_{n=1}^{\infty} b_n = B\right) \\ &= M \text{ (say), which is finite.} \qquad \left(\text{since both } \sum_{n=1}^{N-1} a_n \text{ and } B \text{ are finite.}\right) \end{aligned}$$

$$\Rightarrow A_n \leq M, \ \forall\, n \geq N.$$

As $\{A_n\}$ is non-decreasing, $A_1 \leq A_2 \leq \cdots \leq A_N \leq A_{N+1} \leq \cdots \leq M$.

$$\Rightarrow A_n \leq M, \ \forall n$$

$\Rightarrow \{A_n\}$ is bounded, hence $\sum a_n$ is convergent.

Proof of Part 2

Suppose on the contrary that $\sum b_n$ is convergent and let $\sum b_n = B$.
Consider, for $n \geq N$,

$$\begin{aligned} A_n &= a_1 + a_2 + \cdots + a_{N-1} + a_N + a_{N+1} + \cdots + a_n \\ &\leq (a_1 + a_2 + \cdots + a_{N-1}) + (b_N + b_{N+1} + \cdots + b_n) \\ &= \sum_{n=1}^{N-1} a_n + \sum_{n=N}^{\infty} b_n \leq \sum_{n=1}^{N-1} a_n + \sum_{n=1}^{\infty} b_n \\ &= \sum_{n=1}^{N-1} a_n + B, \qquad \text{(since } \sum_{n=1}^{\infty} b_n = B) \\ &= M \text{ (say), which is finite.} \qquad \text{(since both } \sum_{n=1}^{N-1} a_n \text{ and } B \text{ are finite.)} \end{aligned}$$

$$\Rightarrow A_n \leq M, \ \forall\, n \geq N.$$

As $\{A_n\}$ is non-decreasing, $A_1 \leq A_2 \leq \cdots \leq A_N \leq A_{N+1} \leq \cdots \leq M$.

$$\Rightarrow A_n \leq M, \ \forall n.$$

$\Rightarrow \{A_n\}$ is bounded, hence $\sum a_n$ is convergent, which is a contradiction.

Applying the Comparison Test

Example 70.

The series

$$\sum_{n=1}^{\infty} \frac{5}{5n-1}$$

diverges because its nth term

$$\frac{5}{5n-1} = \frac{1}{n-\frac{1}{5}} > \frac{1}{n}$$

is greater than the nth term of the divergent harmonic series.

Example 71.

The series

$$\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

converges because its terms are all positive and less than or equal to the corresponding terms of

$$1 + \sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots = 3.$$

The fact that 3 is an upper bound for the partial sums of $\sum_{n=0}^{\infty} \frac{1}{n!}$ *does not mean that the series converges to 3. We shall see that the series converges to e.*

Applying the Comparison Test

Example 72.

The series

$$5 + \frac{2}{3} + \frac{1}{7} + 1 + \frac{1}{2+\sqrt{1}} + \frac{1}{4+\sqrt{2}} + \frac{1}{8+\sqrt{3}} + \cdots + \frac{1}{2^n+\sqrt{n}} + \cdots$$

converges. To see this, we ignore the first three terms and compare the remaining terms with those of the convergent geometric series $\sum_{n=0}^{\infty} \frac{1}{2^n}$.

The term $1/(2^n + \sqrt{n})$ *of the truncated sequence is less than the corresponding term* $1/2^n$ *of the geometric series. We see that term by term we have the comparison,*

$$1 + \frac{1}{2+\sqrt{1}} + \frac{1}{4+\sqrt{2}} + \frac{1}{8+\sqrt{3}} + \cdots \le 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

So the truncated series and the original series converge by an application of the Comparison Test.

The Limit Comparison Test

We now introduce a comparison test that is particularly useful for series in which a_n **is a rational function of** n**.**

Theorem 73 (Limit Comparison Test).

Suppose that $a_n > 0$ and $b_n > 0$ for all $n \geq N$ (N an integer)

1. *If* $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, *then* $\sum a_n$ *and* $\sum b_n$ *both converge or both diverge.*
2. *If* $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ *and* $\sum b_n$ *converges, then* $\sum a_n$ *converges.*
3. *If* $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ *and* $\sum b_n$ *diverges, then* $\sum a_n$ *diverges.*

Proof of Part 1

As $\dfrac{a_n}{b_n} \to c$, by definition it follows that given any $\varepsilon > 0$, there exists a positive integer N_1 such that $\left|\dfrac{a_n}{b_n} - c\right| < \varepsilon,\ \forall\ n > N_1$.
In particular, for $\varepsilon = c/2$, there exists a $+$ve integer m such that

$$\begin{aligned} \left|\frac{a_n}{b_n} - c\right| &< \frac{c}{2},\ \forall\ n > N_1 \\ \Rightarrow -\frac{c}{2} < \frac{a_n}{b_n} - c &< \frac{c}{2},\ \forall\ n > N_1 \\ \Rightarrow c - \frac{c}{2} < \frac{a_n}{b_n} &< c + \frac{c}{2},\ \forall\ n > N_1 \\ \Rightarrow \frac{c}{2} < \frac{a_n}{b_n} &< \frac{3c}{2},\ \forall\ n > N_1 \\ \Rightarrow \left(\frac{c}{2}\right) b_n < a_n &< \left(\frac{3c}{2}\right) b_n,\ \forall\ n > N_1. \end{aligned}$$

Thus, by using direct comparison test,
$\left(\dfrac{c}{2}\right) b_n < a_n \Rightarrow$ if $\sum b_n$ diverges, then $\sum a_n$ diverges and
$a_n < \left(\dfrac{3c}{2}\right) b_n \Rightarrow$ if $\sum b_n$ converges, then $\sum a_n$ converges.
Hence, the result follows.

Proof of Part 2

As $\dfrac{a_n}{b_n} \to 0$, for $\varepsilon = 1$, there exists a positive integer N_2 such that

$$\left|\frac{a_n}{b_n} - 0\right| < 1, \ \forall \ n > N_2$$
$$\Rightarrow -1 < \frac{a_n}{b_n} < 1, \ \forall \ n > N_2$$
$$\Rightarrow -b_n < a_n < b_n, \ \forall \ n > N_2$$

Thus, by using direct comparison test, $a_n < b_n \Rightarrow$ if $\sum b_n$ converges, then $\sum a_n$ converges.

Proof of Part 3

As $\dfrac{a_n}{b_n} \to \infty$, by definition, for every real number M, there exists a positive integer N_3 such that $\dfrac{a_n}{b_n} > M, \ \forall \ n > N_3$.

In particular, for $M = 1$, there exists a positive integer N_4 such that

$$\frac{a_n}{b_n} > 1, \ \forall \ n > N_4$$

$$\Rightarrow a_n > b_n, \ \forall \ n > N_4.$$

Thus, by using direct comparison test, if $\sum b_n$ diverges, then $\sum a_n$ also diverges.

Using the Limit Comparison Test

Example 74.

Which of the following series converge, and which diverge?

(a) $\frac{3}{4} + \frac{5}{9} + \frac{7}{16} + \frac{9}{25} + \cdots = \sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2} = \sum_{n=1}^{\infty} \frac{2n+1}{n^2+2n+1}$

(b) $\frac{1}{1} + \frac{1}{3} + \frac{1}{7} + \frac{1}{15} + \cdots = \sum_{n=1}^{\infty} \frac{1}{2^n - 1}$

(c) $\frac{1+2\ln 2}{9} + \frac{1+3\ln 3}{14} + \frac{1+4\ln 4}{21} + \cdots = \sum_{n=1}^{\infty} \frac{1+n\ln n}{n^2+5}$

Solution

(a) Let $a_n = (2n+1)/(n^2+2n+1)$.

For large n, we expect a_n to behave like

$$2n/n^2 = 2/n$$

since the leading terms dominate for large n, so we let $b_n = 1/n$. Since

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n} \quad \text{diverges}$$

and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^2+n}{n^2+2n+1} = 2,$$

$\sum a_n$ diverges by Part 1 of the Limit Comparison Test. We could just as well have taken $b_n = 2/n$, but $1/n$ is simpler.

(b) Let $a_n = 1/(2^n - 1)$. For large n, we expect a_n to behave like $1/2^n$ so we let $b_n = 1/2^n$.

Since

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{2^n} \quad \text{converges}$$

and

$$\begin{aligned} \lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{2^n}{2^n - 1} \\ &= \lim_{n\to\infty} \frac{1}{1 - (1/2^n)} \\ &= 1. \end{aligned}$$

$\sum a_n$ converges by Part 1 of the Limit Comparison Test.

Solution (contd...)

(c) Let $a_n = (1 + n \ln n)/(n^2 + 5)$. For large n, we expect a_n to behave like

$$(n \ln n)/n^2 = (\ln n)/n,$$

which is greater than $1/n$ for $n \geq 3$, so we take $b_n = 1/n$.

Since

$$\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \frac{1}{n} \quad \text{diverges}$$

and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n + n^2 \ln n}{n^2 + 5} = \infty,$$

$\sum a_n$ diverges by Part 3 of the Limit Comparison Test.

Example 75.

Does $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^{3/2}}$ *converge?*

Because $\ln n$ grows more slowly than n^c for any positive constant c, we would expect to have

$$\frac{\ln n}{n^{3/2}} < \frac{n^{1/4}}{n^{3/2}} = \frac{1}{n^{5/4}}$$

for n sufficiently large. Indeed, taking $a_n = (\ln n)/n^{3/2}$ and $b_n = 1/n^{5/4}$, we have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln n}{n^{1/4}} = \lim_{n\to\infty} \frac{1/n}{(1/4)n^{-3/4}} = \lim_{n\to\infty} \frac{4}{n^{1/4}} = 0.$$

Since $\sum b_n = \sum (1/n^{5/4})$ (a p-series with $p > 1$) converges, $\sum a_n$ converges by Part 2 of the Limit Comparison Test.

Exercise 76 (Direct Comparison Test).

Use the comparison test to determine if each series converges or diverges.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n^4+2}$
2. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^{3/2}}$
4. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n3^n}$
5. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{\sqrt{n^2+3}}$

Solution

1. converges
2. diverges
3. converges
4. converges
5. diverges

Exercise 77 (Limit Comparison Test).

Use the limit comparison test to determine if each series converges or diverges.

1. $\displaystyle\sum_{n=1}^{\infty} \sqrt{\frac{n+1}{n^2+2}}$
2. $\displaystyle\sum_{n=2}^{\infty} \frac{n(n+1)}{(n^2+1)(n-1)}$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{3+4^n}$
4. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$
5. $\displaystyle\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n^2}\right)$

Solution

1. diverges
2. diverges
3. converges
4. diverges
5. converges

Exercise 78 (Determining Convergence or Divergence).

Which of the series converge, and which diverge? Give reasons for your answers.

1. $\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n} + \sqrt[3]{n}}$
2. $\sum_{n=1}^{\infty} \frac{sin^2 n}{2^n}$
3. $\sum_{n=1}^{\infty} \frac{n+1}{n^2\sqrt{n}}$
4. $\sum_{n=1}^{\infty} \left(\frac{n}{3n+1}\right)^n$
5. $\sum_{n=3}^{\infty} \frac{1}{\ln(\ln n)}$
6. $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\ln n}$
7. $\sum_{n=2}^{\infty} \frac{\ln(n+1)}{n+1}$
8. $\sum_{n=1}^{\infty} \frac{2^n - n}{n2^n}$
9. $\sum_{n=1}^{\infty} \tan \frac{1}{n}$

Solution

1. diverges when compared with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$
2. converges when compared with $\sum_{n=1}^{\infty} \frac{1}{2^n}$
3. converges when compared with $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$
4. converges when compared with $\sum_{n=1}^{\infty} \frac{1}{3^n}$
5. diverges when compared with $\sum_{n=1}^{\infty} \frac{1}{n}$
6. diverges when compared with $\sum_{n=1}^{\infty} \frac{1}{n}$
7. diverges when compared with $\sum_{n=1}^{\infty} \frac{1}{n}$
8. diverges when compared with $\sum_{n=1}^{\infty} \frac{1}{n}$
9. diverges

Exercise 79 (Determining Convergence or Divergence).

Which of the series converge, and which diverge? Give reasons for your answers.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{sec^{-1} n}{n^{1.3}}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{\coth n}{n^2}$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+2+3+\cdots+n}$
4. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+2^2+3^2+\cdots+n^2}$

Solution

1. converges when compared with $\sum_{n=1}^{\infty} \frac{1}{n^{1.3}}$
2. converges when compared with $\sum_{n=1}^{\infty} \frac{1}{n^2}$
3. converges when compared with $\sum_{n=1}^{\infty} \frac{1}{n^2}$
4. converges when compared with $\sum_{n=1}^{\infty} \frac{1}{n^3}$

Exercise 80.

If $\sum_{n=1}^{\infty} a_n$ *is a convergent series of nonnegative numbers, can anything be said about* $\sum_{n=1}^{\infty} (a_n/n)$*? Explain.*

Solution

Yes, $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges by the Direct Comparison Test because $\frac{a_n}{n} < a_n$.

Exercise 81.

Suppose that $a_n > 0$ *and* $b_n > 0$ *for* $n \geq N$ *(N an integer). If* $\lim_{n \to \infty} (a_n / b_n) = \infty$ *and* $\sum a_n$ *converges, can anything be said about* $\sum b_n$*? Give reasons for your answer.*

Solution

$\lim_{n\to\infty} \frac{a_n}{b_n} = \infty \Rightarrow$ there exists an integer N such that for all $n > N$, $\frac{a_n}{b_n} > 1 \Rightarrow a_n > b_n$.

If $\sum a_n$ converges, then $\sum b_n$ converges by the Direct Comparison Test.

Exercise 82.

Prove that if $\sum a_n$ *is a convergent series of nonnegative terms, then* $\sum a_n^2$ *converges.*

Solution

$\sum a_n$ converges $\Rightarrow \lim_{n\to\infty} a_n = 0 \Rightarrow$ there exists an integer N such that for all $n > N$, $0 \leq a_n < 1 \Rightarrow a_n^2 < a_n \Rightarrow \sum a_n^2$ converges by the Direct Comparison Test.

Exercise 83.

Suppose that $a_n > 0$ *and*

$$\lim_{n\to\infty} n^2 a_n = 0.$$

Prove that $\sum a_n$ *converges.*

Solution

Since $a_n > 0$ and $\lim_{n\to\infty}(n^2 \cdot a_n) = 0$, compare $\sum a_n$ with $\sum \frac{1}{n^2}$, which is a convergent p-series; $\lim_{n\to\infty} \frac{a_n}{1/n^2} = \lim_{n\to\infty}(n^2 \cdot a_n) = 0$.

Hence $\sum a_n$ converges by Limit Comparison Test.

Exercise 84.

Show that

$$\sum_{n=2}^{\infty} \frac{(\ln n)^q}{n^p}$$

converges for $-\infty < q < \infty$ *and* $p > 1$.

(Hint: Limit Comparison with $\sum_{n=2}^{\infty} \frac{1}{n^r}$ *for* $1 < r < p$.*)*

Solution

Let $-\infty < q < \infty$ and $p > 1$. If $q = 0$, then $\sum_{n=2}^{\infty} \frac{(\ln n)^q}{n^p} = \sum_{n=2}^{\infty} \frac{1}{n^p}$, which is a convergent p-series.

If $q \neq 0$, compare with $\sum_{n=2}^{\infty} \frac{1}{n^r}$ where $1 < r < p$, then $\lim_{n\to\infty} \frac{\frac{(\ln n)^q}{n^p}}{1/n^r} = \lim_{n\to\infty} \frac{(\ln n)^q}{n^{p-r}}$, and $p - r > 0$.

If $q < 0 \Rightarrow -q > 0$ and $\lim_{n\to\infty} \frac{(\ln n)^q}{n^{p-r}} = \lim_{n\to\infty} \frac{1}{(\ln n)^{-q} n^{p-r}} = 0$. If $q > 0$,

$\lim_{n\to\infty} \frac{(\ln n)^q}{n^{p-r}} = \lim_{n\to\infty} \frac{q(\ln n)^{q-1}\left(\frac{1}{n}\right)}{(p-r)n^{p-r-1}} = \lim_{n\to\infty} \frac{q(\ln n)^{q-1}}{(p-r)n^{p-r}}$. If $q - 1 \leq 0 \Rightarrow 1 - q \geq 0$ and

$\lim_{n\to\infty} \frac{q(\ln n)^{q-1}}{(p-r)n^{p-r}} = \lim_{n\to\infty} \frac{q}{(p-r)n^{p-r}(\ln n)^{1-q}} = 0$, otherwise, we apply L'Hopital's Rule again.

$\lim_{n\to\infty} \frac{q(q-1)(\ln n)^{q-2}\left(\frac{1}{n}\right)}{(p-r)^2 n^{p-r-1}} = \lim_{n\to\infty} \frac{q(q-1)(\ln n)^{q-2}}{(p-r)^2 n^{p-r}}$. If $q - 2 \leq 0 \Rightarrow 2 - q \geq 0$ and

$\lim_{n\to\infty} \frac{q(q-1)(\ln n)^{q-2}}{(p-r)^2 n^{p-r}} = \lim_{n\to\infty} \frac{q(q-1)}{(p-r)^2 n^{p-r}(\ln n)^{2-q}} = 0$; otherwise, we apply L'Hopital's Rule again.

Since q is finite, there is a positive integer k such that $q - k \leq 0 \Rightarrow k - q \geq 0$. Thus, after k applications of L'Hopital's Rule we obtain

$\lim_{n\to\infty} \frac{q(q-1)\cdots(q-k+1)(\ln n)^{q-k}}{(p-r)^k n^{p-r}} = \lim_{n\to\infty} \frac{q(q-1)\cdots(q-k+1)}{(p-r)^k n^{p-r}(\ln n)^{k-q}} = 0$. Since the limit is 0 in every case,

by Limit Comparison Test, the series $\sum_{n=1}^{\infty} \frac{(\ln n)^q}{n^p}$ converges.

Exercise 85.

Show that

$$\sum_{n=2}^{\infty} \frac{(\ln n)^q}{n^p}$$

diverges for $-\infty < q < \infty$ *and* $0 < p \leq 1$.

(Hint: Limit Comparison with an appropriate p-series)

Solution

Let $-\infty < q < \infty$ and $p \leq 1$. If $q = 0$, then $\sum_{n=2}^{\infty} \frac{(\ln n)^q}{n^p} = \sum_{n=2}^{\infty} \frac{1}{n^p}$, which is a divergent p-series. If $q > 0$, compare with $\sum_{n=2}^{\infty} \frac{1}{n^p}$, which is a divergent p-series. Then $\lim_{n\to\infty} \frac{\frac{(\ln n)^q}{n^p}}{1/n^p} = \lim_{n\to\infty} (\ln n)^q = \infty$. If $q < 0 \Rightarrow -q > 0$, compare with $\sum_{n=2}^{\infty} \frac{1}{n^r}$, where $0 < p < r \leq 1$. $\lim_{n\to\infty} \frac{\frac{(\ln n)^q}{n^p}}{1/n^r} = \lim_{n\to\infty} \frac{(\ln n)^q}{n^{p-r}} = \lim_{n\to\infty} \frac{n^{r-p}}{(\ln n)^{-q}}$ since $r - p > 0$. Apply L'Hopital's to obtain $\lim_{n\to\infty} \frac{(r-p)n^{r-p-1}}{(-q)(\ln n)^{-q-1}\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{(r-p)n^{r-p}}{(-q)(\ln n)^{-q-1}}$. If $-q-1 \leq 0 \Rightarrow q+1 \geq 0$ and $\lim_{n\to\infty} \frac{(r-p)n^{r-p}(\ln n)^{q+1}}{(-q)} = \infty$, otherwise, we apply L'Hopital's Rule again to obtain $\lim_{n\to\infty} \frac{(r-p)^2 n^{r-p-1}}{(-q)(-q-1)(\ln n)^{-q-2}\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{(r-p)^2 n^{r-p}}{(-q)(-q-1)(\ln n)^{-q-2}}$. If $-q-2 \leq 0 \Rightarrow q+2 \geq 0$ and $\lim_{n\to\infty} \frac{(r-p)^2 n^{r-p}}{(-q)(-q-1)(\ln n)^{-q-2}} = \lim_{n\to\infty} \frac{(r-p)^2 n^{r-p}(\ln n)^{q+2}}{(-q)(-q-1)} = \infty$, otherwise, we apply L'Hopital's Rule again. Since q is finite, there is a positive integer k such that $-q-k \leq 0 \Rightarrow q+k \geq 0$. Thus, after k applications of L'Hopital's Rule we obtain

$\lim_{n\to\infty} \frac{(r-p)^k n^{r-p}}{(-q)(-q-1)\cdots(-q-k+1)(\ln n)^{-q-k}} = \lim_{n\to\infty} \frac{(r-p)^k n^{r-p}(\ln n)^{q+k}}{(-q)(-q-1)\cdots(-q-k+1)} = \infty$. Since the limit is ∞ if $q > 0$ or if $q < 0$ and $p < 1$, by Limit comparison test, the series $\sum_{n=1}^{\infty} \frac{(\ln n)^q}{n^{p-r}}$ diverges. Finally if $q < 0$ and $p = 1$ then $\sum_{n=2}^{\infty} \frac{(\ln n)^q}{n^p} = \sum_{n=2}^{\infty} \frac{(\ln n)^q}{n}$. Compare with $\sum_{n=2}^{\infty} \frac{1}{n}$, which is a divergent p-series. For $n \geq 3$, $\ln n \geq 1 \Rightarrow (\ln n)^q \geq 1 \Rightarrow \frac{(\ln n)^q}{n} \geq \frac{1}{n}$. Thus $\sum_{n=2}^{\infty} \frac{(\ln n)^q}{n}$ diverges by Comparison Test. Thus, if $-\infty < q < \infty$ and $p \leq 1$, the series $\sum_{n=1}^{\infty} \frac{(\ln n)^q}{n^{p-r}}$ diverges.

Exercise

Exercise 86.

Use results of the above two exercises to determine if each series converges or diverges.

(a) $\displaystyle\sum_{n-2}^{\infty} \frac{(\ln n)^3}{n^4}$

(b) $\displaystyle\sum_{n-2}^{\infty} \sqrt{\frac{\ln n}{n}}$

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{(\ln n)^{1/5}}{n^{0.99}}$

(d) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n. \ln n}}$

Solution

1. Converges by Exercise 84 with $q = 3$ and $p = 4$.
2. Diverges by Exercise 85 with $q = \frac{1}{2}$ and $p = \frac{1}{2}$.
3. Diverges by Exercise 85 with $q = \frac{1}{5}$ and $p = 0.99$.
4. Diverges by Exercise 85 with $q = -\frac{1}{2}$ and $p = \frac{1}{2}$.

Exercise 87 (Decimal numbers).

Any real number in the interval $[0, 1]$ *can be represented by a decimal (not necessarily unique) as*

$$0 \cdot d_1 d_2 d_3 d_4 \ldots = \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \frac{d_4}{10^4} + \cdots ,$$

where d_i *is one of the integers* $0, 1, 2, 3, \ldots, 9$.
Prove that the series on the right-hand side always converges.

Exercise 88.

1. *If* $\sum a_n$ *is a convergent series of positive terms, prove that* $\sum \sin(a_n)$ *converges.*
2. *Show that* $\sum_{n=1}^{\infty} \left[\frac{\sqrt{n+1}-\sqrt{n}}{n^p}\right]$ *converges for* $p > \frac{1}{2}$ *and diverges for* $0 < p \leq \frac{1}{2}$.

Absolute Convergence

Absolutely Convergent

When some of the terms of a series are positive and others are negative, the series may or may not converge. For example, the geometric series

$$5-\frac{5}{4}+\frac{5}{16}-\frac{5}{64}+\cdots=\sum_{n=0}^{\infty}5\left(\frac{-1}{4}\right)^{n} \tag{4}$$

converges (since $|r|=\frac{1}{4}<1$), whereas the different geometric series

$$1-\frac{5}{4}+\frac{25}{16}-\frac{125}{64}+\cdots=\sum_{n=0}^{\infty}\left(\frac{-5}{4}\right)^{n} \tag{5}$$

diverges (since $|r|=5/4>1$). In series (4), there is some cancelation in the partial sums, which may be assisting the convergence property of the series.

Absolutely Convergent

However, if we make all of the terms positive in series (4) to form the new series

$$5 + \frac{5}{4} + \frac{5}{16} + \frac{5}{64} + \cdots = \sum_{n=0}^{\infty} \left| 5\left(\frac{-1}{4}\right)^n \right| = \sum_{n=0}^{\infty} 5\left(\frac{1}{4}\right)^n,$$

we see that it still converges. For a general series with both positive and negative terms, we can apply the tests for convergence studied before to the series of absolute values of its terms.

In doing so, we are led naturally to the following concept.

Definition 89 (Absolutely Convergent).

A series $\sum a_n$ converges absolutely *(is* absolutely convergent*) if the corresponding series of absolute values,* $\sum |a_n|$, *converges.*

Absolutely Convergent

The geometric series

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$$

converges absolutely because the corresponding series of absolute values

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

converges. The series

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots = \sum_{n=1}^{\infty} \left(\frac{-1}{2}\right)^n$$

is also convergent. The situation is always true : An absolutely convergent series is convergent as well, which will be proved now.

The Absolute Convergence Test

Theorem 90 (The Absolute Convergence Test).

If $\sum_{n=1}^{\infty} |a_n|$ *converges, then* $\sum_{n=1}^{\infty} a_n$ *converges.*

Proof of The Absolute Convergence Test

For each n, $-|a_n| \leq a_n \leq |a_n|$, so $0 \leq a_n + |a_n| \leq 2|a_n|$. If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} 2|a_n|$ converges and, by the Direct Comparison Test, the nonnegative series $\sum_{n=1}^{\infty}(a_n + |a_n|)$converges. The equality $a_n = (a_n + |a_n|) - |a_n|$ now lets us express $\sum_{n=1}^{\infty} a_n$ as the difference of two convergent series :

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}(a_n + |a_n| - |a_n|) = \sum_{n=1}^{\infty}(a_n + |a_n|) - \sum_{n=1}^{\infty} |a_n|.$$

Therefore, $\sum_{n=1}^{\infty} a_n$ converges.

Example 91.

For

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots$$

the corresponding series of absolute values is convergent series

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots .$$

The original series converges because it converges absolutely.

Example 92.

For

$$\sum_{n=1}^{\infty} \frac{\sin n}{n^2} = \frac{\sin 1}{1} + \frac{\sin 2}{4} + \frac{\sin 3}{9} + \cdots$$

the corresponding series of absolute values is

$$\sum_{n=1}^{\infty} \left| \frac{\sin n}{n^2} \right| = \left| \frac{\sin 1}{1} \right| + \left| \frac{\sin 2}{4} \right| + \cdots$$

which converges by comparison with $\sum_{n=1}^{\infty} (1/n^2)$ *because* $|\sin n| \leq 1$ *for every* n*. The original series converges absolutely; therefore it converges.*

Ratio and Root Tests

Ratio Test

The Ratio Test measures the rate of growth (or decline) of a series by the ratio

$$\frac{a_{n+1}}{a_n}.$$

For a geometric series $\sum ar^n$, this rate is constant

$$\frac{ar^{n+1}}{ar^n} = r,$$

and the series converges if and only if its ratio is less than 1 in absolute value.

The Ratio Test is a powerful rule extending that result.

We shall now discuss convergence of series using the "Ratio and Root Tests."

Theorem 93 (Ratio Test).

Let $\sum a_n$ *be any series and suppose that*

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \rho.$$

Then

(a) *The series converges absolutely if* $\rho < 1$.

(b) *The series diverges if* $\rho > 1$ *or* ρ *is infinite.*

(c) *The test is inconclusive if* $\rho = 1$.

Proof of Ratio Test

(a) **Case :** $\rho < 1$

Let r be a number between ρ and 1. Then the number $\varepsilon = r - \rho$ is positive.

Since $\left|\dfrac{a_{n+1}}{a_n}\right| \to \rho$, $\left|\frac{a_{n+1}}{a_n}\right|$ must lie within ε of ρ, when n is large enough, say for all $n \geq N$. In particular $\left|\frac{a_{n+1}}{a_n}\right| < \rho + \varepsilon = r$, when $n \geq N$. That is,

$$\begin{aligned}
|a_{N+1}| &< r|a_N| \\
|a_{N+2}| &< r|a_{N+1}| < r^2|a_N| \\
|a_{N+3}| &< r|a_{N+2}| < r^3|a_N| \\
&\vdots \\
|a_{N+m}| &< r|a_{N+m-1}| < r^m|a_N|.
\end{aligned}$$

Proof of Ratio Test (contd...)

Therefore,

$$\sum_{m=N}^{\infty} |a_m| = \sum_{m=0}^{\infty} |a_{N+m}| \leq \sum_{m=0}^{\infty} |a_N| r^m = |a_N| \sum_{m=0}^{\infty} r^m.$$

The geometric series on the right-hand side converges because $0 < r < 1$, so the series of absolute values $\sum_{m=N}^{\infty} |a_m|$ converges by the Direct Comparison Test.

Because adding or deleting finitely many terms in a series does not affect its convergence or divergence property, the series $\sum_{n=1}^{\infty} |a_n|$ also converges.

That is, the series $\sum a_n$ is absolutely convergent.

Proof of Ratio Test (contd...)

(b) **Case :** $1 < \rho \leq \infty$

From some index M on, $\left|\frac{a_{N+1}}{a_n}\right| > 1$ and $|a_M| < |a_{M+1}| < |a_{M+2}| < \cdots$.

The terms of the series do not approach zero as n becomes infinite, and the series diverges by the nth Term Test.

Proof of Ratio Test (contd...)

(c) **Case :** $\rho = 1$

The two series

$$\sum_{n=1}^{\infty} \frac{1}{n} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^2}$$

show that some other test for convergence must be used when $\rho = 1$.

For $\sum_{n=1}^{\infty} \frac{1}{n}$: $\quad \left|\frac{a_{n+1}}{a_n}\right| = \frac{1/(n+1)}{1/n} = \frac{n}{n+1} \to 1.$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$: $\quad \left|\frac{a_{n+1}}{a_n}\right| = \frac{1/(n+1)^2}{1/n^2} = \left(\frac{n}{n+1}\right)^2 \to 1^2 = 1.$

In both cases, $\rho = 1$, yet the first series diverges, whereas the second converges.

Ratio Test

The Ratio Test is often effective when the terms of a series contain factorials of expressions involving n or expressions raised to a power involving n.

Example 94 (Applying The Ratio Test).

Investigate the convergence of the following series.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n + 5}{3^n}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n!n!}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{4^n n!n!}{(2n)!}$.

Solution

(a) For the series $\sum_{n=0}^{\infty} (2^n + 5)/3^n$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(2^{n+1}+5)/3^{n+1}}{(2^n+5)/3^n} = \frac{1}{3}.\frac{2^{n+1}+5}{2^n+5} = \frac{1}{3}.\left(\frac{2+5.2^{-n}}{1+5.2^{-n}}\right) \to \frac{1}{3}.\frac{2}{1} = \frac{2}{3}.$$

The series converges because $\rho = 2/3$ is less than 1. This does not mean that $2/3$ is the sum of the series. In fact,

$$\sum_{n=0}^{\infty} \frac{2^n+5}{3^n} = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n + \sum_{n=0}^{\infty} \frac{5}{3^n} = \frac{1}{1-(2/3)} + \frac{5}{1-(1/3)} = \frac{21}{2}.$$

Solution (contd...)

(b) If $a_n = \frac{(2n)!}{n!n!}$, then $a_{n+1} = \frac{(2n+2)!}{(n+1)!(n+1)!}$ and

$$\begin{aligned} \left|\frac{a_{n+1}}{a_n}\right| &= \frac{n!n!(2n+2)(2n+1)(2n)!}{(n+1)!(n+1)!(2n)!} \\ &= \frac{(2n+2)(2n+1)}{(n+1)(n+1)} \\ &= \frac{4n+2}{n+1} \to 4. \end{aligned}$$

The series diverges because $\rho = 4$ is greater than 1.

Solution (contd...)

(c) If $a_n = 4^n n!n!/(2n)!$, then

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{4^{n+1}(n+1)!(n+1)!}{(2n+2)(2n+1)(2n)!} \cdot \frac{(2n)!}{4^n n!n!}$$
$$= \frac{4(n+1)(n+1)}{(2n+2)(2n+1)} = \frac{2(n+1)}{2n+1} \to 1.$$

Because the limit is $\rho = 1$, we cannot decide from the Ratio Test whether the series converges. When we notice that $a_{n+1}/a_n = (2n+2)/(2n+1)$, we conclude that a_{n+1} is always greater than a_n because $(2n+2)/(2n+1)$ is always greater than 1.

Therefore, all terms are greater than or equal to $a_1 = 2$, and the nth term does not approach zero as $n \to \infty$. The series diverges.

Problems using (D'Alembert's) Ratio Test

Example 95.

Discuss the convergence of the following series :

1. $1 + \frac{2!}{2^2} + \frac{3!}{3^3} + \frac{4!}{4^4} + \dots$ *[Ans: Convergent]*
2. $1 + \frac{2^p}{2!} + \frac{3^p}{3!} + \frac{4^p}{4!} + \dots \ (p > 0)$ *[Ans: Convergent]*
3. $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{9} + \dots$ *[Ans: Convergent]*
4. $1 + \frac{x}{2} + \frac{x^2}{5} + \frac{x^3}{10} + \dots + \frac{x^n}{n^2+1} + \dots$
   [Ans: Convergent if $x \leq 1$ and divergent if $x > 1$]
5. $1 + 3x + 5x^2 + 7x^3 + 9x^4 + \dots$
   [Ans: Convergent if $x < 1$ and divergent if $x \geq 1$]

The convergence tests we have so far for $\sum a_n$ work best when the formula for a_n is relatively simple. But consider the following.

Example 96.

Let $a_n = \begin{cases} n/2^n & n \text{ odd} \\ 1/2^n & n \text{ even.} \end{cases}$

Does $\sum a_n$ *converge?*

Solution

To investigate the convergence, we write out several terms of the series :

$$\sum_{n=1}^{\infty} a_n = \frac{1}{2^1} + \frac{1}{2^2} + \frac{3}{2^3} + \frac{1}{2^4} + \frac{5}{2^5} + \frac{1}{2^6} + \frac{7}{2^7} + \cdots$$

$$= \frac{1}{2} + \frac{1}{4} + \frac{3}{8} + \frac{1}{16} + \frac{5}{32} + \frac{1}{64} + \frac{7}{128} + \cdots$$

Clearly, this is not a geometric series.

The nth term approaches zero as $n \to \infty$, so we do not know if the series diverges.

The Integral Test does not look promising.

Solution (contd...)

The Ratio Test produces

$$\frac{a_{n+1}}{a_n} = \begin{cases} \frac{1}{2n} & \text{n odd} \\ \frac{n+1}{2} & \text{n even} \end{cases}$$

As $n \to \infty$, the ratio is alternately small and large and has no limit.

A test that will answer the question (the series converges) is the **Root Test**.

The (Cauchy's) Root Test

Theorem 97 (The Root Test).

Let $\sum a_n$ *be any series and suppose that*

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \rho.$$

Then

(a) *the series converges absolutely if* $\rho < 1$,

(b) *the series diverges if* $\rho > 1$ *or* ρ *is infinite,*

(c) *the test is inconclusive if* $\rho = 1$.

Proof of Root Test

(a) **Case :** $\rho < 1$

Choose an $\varepsilon > 0$ so small that $\rho + \varepsilon < 1$. Since $\sqrt[n]{|a_n|} \to \rho$, the terms $\sqrt[n]{|a_n|}$ eventually get closer than ε to ρ.

In other words, there exists an index $M \geq N$ such that $\sqrt[n]{|a_n|} < \rho + \varepsilon$ when $n \geq M$. Then it is also true that $|a_n| < (\rho + \varepsilon)^n$ for $n \geq M$.

Now, $\sum_{n=M}^{\infty} (\rho + \varepsilon)^n$, a geometric series with ratio $(\rho + \varepsilon) < 1$ and theorefore converges. By direct comparison test, $\sum_{n=M}^{\infty} |a_n|$ converges, from which it follows that

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + \cdots + |a_{M-1}| + \sum_{n=M}^{\infty} |a_n|$$

converges. Theorefore $\sum a_n$ converges absolutely.

Proof of Root Test

(b) **Case :** $1 < \rho \leq \infty$

For all indices beyond some integer M. we have $\sqrt[n]{|a_n|} > 1$, so that $|a_n| > 1$ for $n > M$.

The terms of the series do not converge to zero. The series diverges by the nth-Term Test.

(c) **Case :** $\rho = 1$

The series $\sum_{n=1}^{\infty}(1/n)$ and $\sum_{n=1}^{\infty}(1/n^2)$ show that the test is not conclusive when $\rho = 1$.

The first series diverges and the second converges, but in both cases $\sqrt[n]{|a_n|} \to 1$.

Problems using (Cauchy's) n^{th}-root Test

Example 98.

Discuss the convergence/divergence of the following series :

1. $\sum \left(\frac{n}{n+1}\right)^{n^2} \left(or \sum \left(1+\frac{1}{n}\right)^{-n^2} \right)$ *[Ans: Convergent]*
2. $\sum \frac{1}{(\log n)^n}$ *[Ans: Convergent]*
3. $\sum \frac{x^n}{n^n}$ *[Ans: Convergent for all x]*
4. $\sum e^{\sqrt{n}} r^n \; (r > 0)$ *[Ans: Conv. if $0 < r < 1$ & Div. if $r \geq 1$]*
5. $\sum (n^{1/n} - 1)^n$ *[Ans: Convergent]*
6. $\sum n^k x^n$ *[Ans: Convergent if $x < 1$ or $x = 1$ and $k < -1$ and Divergent if $x > 1$ or $x = 1$ and $k \geq -1$]*
7. $\frac{1^3}{3} + \frac{2^3}{3^2} + 1 + \frac{4^3}{3^4} + \cdots$ *[Try this problem using Ratio test also.]* *[Ans: Convergent]*

Example 99 (Applying the Root Test).

Which of the following series converges, and which diverges?

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^2}$

(c) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$

Solution

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges because $\sqrt[n]{\frac{n^2}{2^n}} = \frac{\sqrt[n]{n^2}}{\sqrt[n]{2^n}} = \frac{(\sqrt[n]{n})^2}{2} \to \frac{1}{2} < 1$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^2}$ diverges because $\sqrt[n]{\frac{n^2}{2^n}} = \frac{2}{(\sqrt[n]{n})^2} \to \frac{2}{1} > 1.$

(c) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$ converges because $\sqrt[n]{\left(\frac{1}{1+n}\right)^n} = \frac{1}{1+n} \to 0 < 1.$

Example 100.

Let $a_n = \begin{cases} n/2^n & n \text{ odd} \\ 1/2^n & n \text{ even.} \end{cases}$ *Does* $\sum a_n$ *converge?*

Solution

We apply the Root Test, finding that

$$\sqrt[n]{|a_n|} = \begin{cases} \sqrt[n]{n/2}, & \text{n odd} \\ 1/2, & \text{n even.} \end{cases} \tag{6}$$

Therefore, $\frac{1}{2} \leq \sqrt[n]{|a_n|} \leq \frac{\sqrt[n]{n}}{2}$. Since $\sqrt[n]{n} \to 1$, we have $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1/2$ by the Sandwich Theorem.

The limit is less than 1, so the series converges by the Root Test.

Exercise 101 (Determining Convergence or Divergence).

Which of the series converge, and which diverge? Give reasons for your answers. (When checking your answers, remember there may be more than one way to determine a series' convergence or divergence.)

1. $\sum_{n=1}^{\infty}(-1)^n n^2 e^{-n}$
2. $\sum_{n=1}^{\infty} n!(-e)^{-n}$
3. $\sum_{n=1}^{\infty}\left(\frac{n-2}{n}\right)^n$
4. $\sum_{n=1}^{\infty}\sin^n\left(\frac{1}{\sqrt{n}}\right)$
5. $\sum_{n=1}^{\infty}(-1)^n\left(1-\frac{1}{3n}\right)^n$
6. $\sum_{n=1}^{\infty}\frac{(-\ln n)^n}{n^n}$

Solution

1. converges by the Ratio Test:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\left(\frac{(n+1)^2}{e^{n+1}}\right)}{\left(\frac{n^2}{e^n}\right)} = \lim_{n\to\infty} \frac{(n+1)^2}{e^{n+1}} \cdot \frac{e^n}{n^2} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^2 \left(\frac{1}{e}\right) = \frac{1}{e} < 1.$$

2. diverges by the Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\left(\frac{(n+1)!}{e^{n+1}}\right)}{\left(\frac{n!}{e^n}\right)} = \lim_{n\to\infty} \frac{(n+1)!}{e^{n+1}} \cdot \frac{e^n}{n!} = \lim_{n\to\infty} \frac{n+1}{e} = \infty.$

3. diverges; $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{n-2}{n}\right)^n = \lim_{n\to\infty} \left(1+\frac{-2}{n}\right)^n = e^{-2} \neq 0$

4. $\left[\sin\left(\frac{1}{\sqrt{n}}\right)\right]^n \geq 0$ for all $n \geq 1$; $\lim_{n\to\infty} \sqrt[n]{\left[\sin\left(\frac{1}{\sqrt{n}}\right)\right]^n} = \lim_{n\to\infty} \sin\left(\frac{1}{\sqrt{n}}\right) = \sin(0) = 0 < 1 \Rightarrow \sum_{n=1}^{\infty} \left[\sin\left(\frac{1}{\sqrt{n}}\right)\right]^n$ converges.

5. diverges; $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(1-\frac{1}{3n}\right)^n = \lim_{n\to\infty} \left(1+\frac{\left(-\frac{1}{3}\right)}{n}\right)^n = e^{-1/3} \approx 0.72 \neq 0$

6. converges by the nth-Root Test:

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{(\ln n)^n}{n^n}} = \lim_{n\to\infty} \frac{((\ln n)^n)^{1/n}}{(n^n)^{1/n}} = \lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{\left(\frac{1}{n}\right)}{1} = 0 < 1$$

Exercise 102 (Determining Convergence or Divergence).

Which of the series converge, and which diverge? Give reasons for your answers. (When checking your answers, remember there may be more than one way to determine a series' convergence or divergence.)

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(n+1)(n+2)}{n!}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{n2^n(n+1)!}{3^n n!}$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(2n+1)!}$
4. $\displaystyle\sum_{n=2}^{\infty} \frac{n}{(\ln n)^{(n/2)}}$
5. $\displaystyle\sum_{n=1}^{\infty} \frac{n! \ln n}{n(n+2)!}$

Solution

1. converges by the Ratio Test:
$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+2)(n+3)}{(n+1)!} \cdot \frac{n!}{(n+1)(n+2)} = 0 < 1$$
2. converges by the Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} =$
$$\lim_{n\to\infty} \frac{(n+1)2^{n+1}(n+2)!}{3^{n+1}(n+1)!} \cdot \frac{3^n n!}{n2^n(n+1)!} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)\left(\frac{2}{3}\right)\left(\frac{n+2}{n+1}\right) = \frac{2}{3} < 1$$
3. converges by the Ratio Test:
$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!}{(2n+3)!} \cdot \frac{(2n+1)!}{n!} = \lim_{n\to\infty} \frac{n+1}{(2n+3)(2n+2)} = 0 < 1$$
4. converges by the Root Test:
$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{n}{(\ln n)^{n/2}}} = \lim_{n\to\infty} \frac{\sqrt[n]{n}}{\sqrt{\ln n}} = \frac{\lim_{n\to\infty} \sqrt[n]{n}}{\lim_{n\to\infty} \sqrt{\ln n}} = 0 < 1$$
$$\left(\lim_{n\to\infty} \sqrt[n]{n} = 1\right)$$
5. converges by the Direct Comparison Test:
$\frac{n! \ln n}{n(n+2)!} = \frac{\ln n}{n(n+1)(n+2)} < \frac{n}{n(n+1)(n+2)} = \frac{1}{(n+1)(n+2)} < \frac{1}{n^2}$ which is the nth-term of a convergent p-series.

Exercise 103 (Recursively Defined Terms).

Which of the series $\sum_{n=1}^{\infty} a_n$ *defined by the formulas converge, and which diverge? Give reasons for your answers.*

1. $a_1 = 2, a_{n+1} = \frac{1+\sin n}{n} a_n$
2. $a_1 = 1, a_{n+1} = \frac{1+\tan^{-1} n}{n} a_n$
3. $a_1 = 3, a_{n+1} = \frac{n}{n+1} a_n$
4. $a_1 = 5, a_{n+1} = \frac{\sqrt[n]{n}}{2} a_n$
5. $a_1 = \frac{1}{2}, a_{n+1} = \frac{n+\ln n}{n+10} a_n$
6. $a_1 = \frac{1}{2}, a_{n+1} = (a_n)^{n+1}$
7. $a_n = \frac{2^n n! n!}{(2n)!}$

Solution

1. converges by the Ratio Test: $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \frac{\left(\frac{1+\sin n}{n}\right)a_n}{a_n} = 0 < 1$

2. converges by the Ratio Test: $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \frac{\left(\frac{1+\tan^{-1} n}{n}\right)a_n}{a_n} = \lim\limits_{n\to\infty} \frac{1+tan^{-1}n}{n} = 0$ since the numerator approaches $1 + \frac{\pi}{2}$ while the denominator tends to ∞.

3. diverges; $a_{n+1} = \frac{n}{n+1} a_n \Rightarrow a_{n+1} = \left(\frac{n}{n+1}\right)\left(\frac{n-1}{n} a_{n-1}\right) \Rightarrow a_{n+1} = \left(\frac{n}{n+1}\right)\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n-1} a_{n-2}\right) \Rightarrow a_{n+1} = \left(\frac{n}{n+1}\right)\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n-1}\right)\cdots\left(\frac{1}{2}\right) a_1 \Rightarrow a_{n+1} = \frac{a_1}{n+1} \Rightarrow a_{n+1} = \frac{3}{n+1}$, which is a constant times the general term of the diverging harmonic series.

4. converges by the Ratio Test: $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \frac{\sqrt[n]{n}}{2} = \frac{1}{2} < 1.$

5. $\frac{n+\ln n}{n+10} > 0$ and $a_1 = \frac{1}{2} \Rightarrow a_n > 0$; $\ln n > 10$ for $n > e^{10} \Rightarrow n + \ln n > n + 10 \Rightarrow \frac{n+\ln n}{n+10} > 1 \Rightarrow a_{n+1} = \frac{n+\ln n}{n+10} a_n > a_n$; thus $a_{n+1} > a_n \geq \frac{1}{2} \Rightarrow \lim\limits_{n\to\infty} a_n \neq 0$, so the series diverges by the nth-Term Test.

6. converges by the Direct Comparison Test: $a_1 = \frac{1}{2}$, $a_2 = \left(\frac{1}{2}\right)^2, a_3 = \left(\left(\frac{1}{2}\right)^2\right)^3 = \left(\frac{1}{2}\right)^6, a_4 = \left(\left(\frac{1}{2}\right)^6\right)^4 = \left(\frac{1}{2}\right)^{24}, \cdots \Rightarrow a_n = \left(\frac{1}{2}\right)^{n!} < \left(\frac{1}{2}\right)^n$ which is the nth-term of a convergent geometric series.

7. converges by the Ratio Test:
$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^{n+1}(n+1)!(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{2^n n! n!} = \lim_{n\to\infty} \frac{2(n+1)(n+1)}{(2n+2)(2n+1)} = \lim_{n\to\infty} \frac{n+1}{2n+1} = \frac{1}{2} < 1.$$

Exercise 104 (Determining Convergence or Divergence).

Which of the series converge, and which diverge? Give reasons for your answers.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^n}{(n^n)^2}$
2. $\displaystyle\lim_{n\to\infty} \frac{n^n}{(2^n)^2}$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot \cdots \cdot (2n-1)}{4^n 2^n n!}$
4. $\displaystyle\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot \cdots \cdot (2n-1)}{[2 \cdot 4 \cdot \cdots \cdot (2n)](3^n+1)}$

Solution

1. diverges by the Root Test:
$$\lim_{n\to\infty} \sqrt[n]{a_n} \equiv \lim_{n\to\infty} \sqrt[n]{\frac{(n!)^n}{(n^n)^2}} = \lim_{n\to\infty} \frac{n!}{n^2} = \infty > 1.$$
2. diverges by the Root Test:
$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{n^n}{(2^n)^2}} = \lim_{n\to\infty} \frac{n}{4} = \infty > 1.$$
3. converges by the Ratio Test: $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} =$
$$\lim_{n\to\infty} \frac{1\cdot 3\cdot\cdots\cdot(2n-1)(2n+1)}{4^{n+1}2^{n+1}(n+1)!} \cdot \frac{4^n 2^n n!}{1\cdot 3\cdot\cdots\cdot(2n-1)} = \lim_{n\to\infty} \frac{2n+1}{(4\cdot 2)(n+1)} = \frac{1}{4} < 1.$$
4. converges by the Ratio Test:
$$a_n = \frac{1\cdot 3\cdots(2n-1)}{(2\cdot 4\cdots 2n)(3^n+1)} = \frac{1\cdot 2\cdot 3\cdot 4\cdots(2n-1)(2n)}{(2\cdot 4\cdots 2n)^2(3^n+1)} = \frac{(2n)!}{(2^n n!)^2(3^n+1)} \Rightarrow$$
$$\lim_{n\to\infty} \frac{(2n+2)!}{[2^{n+1}(n+1)!]^2(3^{n+1}+1)} \cdot \frac{(2^n n!)^2(3^n+1)}{(2n)!} = \lim_{n\to\infty} \frac{(2n+1)(2n+2)(3^n+1)}{2^2(n+1)^2(3^{n+1}+1)}$$
$$= \lim_{n\to\infty} \left(\frac{4n^2+6n+2}{4n^2+8n+4}\right) \frac{(1+3^{-n})}{(3+3^{-n})} = 1 \cdot \frac{1}{3} = \frac{1}{3} < 1.$$

Exercise 105.

Neither the Ratio nor the Root Test helps with p-series. Try them on

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

and show that both tests fail to provide information about convergence.

Solution

Ratio Test : $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{(n+1)^p} \cdot \frac{n^p}{1} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^p = 1^p = 1 \Rightarrow$ no conclusion.

Root Test : $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n^p}} = \lim_{n\to\infty} \frac{1}{(\sqrt[n]{n})^p} = \frac{1}{(1)^p} = 1 \Rightarrow$ no conclusion.

Exercise 106.

Show that neither the Ratio Test nor the Root Test provides information about the convergence of

$$\sum_{n=2}^{\infty} \frac{1}{(\ln n)^p}$$

where p is a constant.

Solution

Ratio Test : $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{(\ln(n+1))^p} \cdot \frac{(\ln n)^p}{1} = \left[\lim_{n\to\infty} \frac{\ln n}{\ln(n+1)}\right]^p =$
$\left[\lim_{n\to\infty} \frac{\left(\frac{1}{n}\right)}{\left(\frac{1}{n+1}\right)}\right]^p = \left(\lim_{n\to\infty} \frac{n+1}{n}\right)^p = (1)^p = 1 \Rightarrow$ no conclusion.

Root Test : $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{(\ln n)^p}} = \frac{1}{\left(\lim_{n\to\infty} (\ln n)^{1/n}\right)^p}$; let
$f(n) = (\ln n)^{1/n}$, then $\ln f(n) = \frac{\ln(\ln n)}{n} \Rightarrow \lim_{n\to\infty} \ln f(n) = \lim_{n\to\infty} \frac{\ln(\ln n)}{n} =$
$\lim_{n\to\infty} \frac{\left(\frac{1}{n \ln n}\right)}{1} = \lim_{n\to\infty} \frac{1}{n \ln n} = 0 \Rightarrow \lim_{n\to\infty} (\ln n)^{1/n} = \lim_{n\to\infty} e^{\ln f(n)} = e^0 = 1;$
therefore $\lim_{n\to\infty} \sqrt[n]{a_n} = \frac{1}{\left(\lim_{n\to\infty} (\ln n)^{1/n}\right)^p} = \frac{1}{(1)^p} = 1 \Rightarrow$ no conclusion.

Observation

Cauchy's n^{th}-root test is more general than D'Alembert's ratio test.

The reason is,

1. $$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} \text{ exists} \Rightarrow \lim_{n\to\infty} (a_n)^{1/n} \text{ exists}$$ (and both the limits are equal, in such a case).

2. $$\lim_{n\to\infty} (a_n)^{1/n} \text{ exists} \nRightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \text{ exists.}$$

Therefore, whenever Ratio Test is applicable, Root Test is also applicable, but not conversely.

In other words, Ratio test may fail but root test may work.

Example 107.

Discuss the convergence/divergence of the following series.

$$a_n = \begin{cases} 2^{-n}; & \text{if } n \text{ is odd} \\ 2^{-n+2}; & \text{if } n \text{ is even} \end{cases}$$

Proof:

- If n is even, then
$$\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} (2^{-n})^{1/n} = \frac{1}{2}.$$
- If n is odd, then
$$\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} (2^{-n+2})^{1/n} = \lim_{n\to\infty} 2^{-1+2/n} = \frac{1}{2}.$$

Thus, in both the cases, $\lim_{n\to\infty} (a_n)^{1/n} = \frac{1}{2} < 1, \ \forall\, n$. Hence, by Cauchy's Root Test, the given series is convergent.

In the above problem, note that Ratio Test fails, but Root Test holds good.

Exercise 108.

Let $a_n = \begin{cases} n/2^n, & \text{if } n \text{ is a prime number} \\ 1/2^n, & \text{otherwise.} \end{cases}$

Does $\sum a_n$ *converge? Give reasons for your answer.*

Solution

$a_n \leq \frac{n}{2^n}$ for every n and the series $\sum_{n=1}^{\infty} \frac{n}{2^n}$ converges by the Ratio Test since $\lim_{n\to\infty} \frac{(n+1)}{2^{n+1}} \cdot \frac{2^n}{n} = \frac{1}{2} < 1$

$\Rightarrow \sum_{n=1}^{\infty} a_n$ converges by the Direct Comparison Test.

Exercise 109.

Show that

$$\sum_{n=1}^{\infty} \frac{2^{(n^2)}}{n!}$$

diverges. Recall from the Laws of Exponents that $2^{(n^2)} = (2^n)^n$.

Solution

$\frac{2^{n^2}}{n!} > 0$ for all $n \geq 1$;

$$\lim_{n\to\infty}\left(\frac{\frac{2^{(n+1)^2}}{(n+1)!}}{\frac{2^{n^2}}{n!}}\right) = \lim_{n\to\infty}\left(\frac{2^{n^2+2n+1}}{(n+1)\cdot n!}\cdot\frac{n!}{2^{n^2}}\right) = \lim_{n\to\infty}\left(\frac{2^{2n+1}}{n+1}\right) = \lim_{n\to\infty}\left(\frac{2\cdot 4^n}{n+1}\right) =$$

$$\lim_{n\to\infty}\left(\frac{2\cdot 4^n \ln 4}{1}\right) = \infty > 1$$

$\Rightarrow \sum_{n=1}^{\infty} \frac{2^{n^2}}{n!}$ diverges.

Exercise 110.

Assume that $\{b_n\}$ is a sequence of positive numbers converging to 4/5. Determine if the following series converge or diverge.

1. $\sum_{n=1}^{\infty} (b_n)^{1/n}$
2. $\sum_{n=1}^{\infty} \left(\frac{5}{4}\right)^n (b_n)$
3. $\sum_{n=1}^{\infty} (b_n)^n$
4. $\sum_{n=1}^{\infty} \frac{1000^n}{n!+b_n}$

Exercise 111.

Assume that $\{b_n\}$ *is a sequence of positive numbers converging to* $1/3$*. Determine if the following series converge or diverge.*

1. $\sum_{n=1}^{\infty} \frac{b_{n+1} b_n}{n 4^n}$
2. $\sum_{n=1}^{\infty} \frac{n^n}{n! b_1^2 b_2^2 \cdots b_n^2}$

References

1. M.D. Weir, J. Hass and F.R. Giordano, Thomas' Calculus, 11th Edition, Pearson Publishers.
2. N. Piskunov, Differential and Integral Calculus, Vol I & II (Translated by George Yankovsky).
3. S.C. Malik and Savitha Arora, Mathematical Analysis, New Age Publishers.
4. R. G. Bartle, D. R. Sherbert, Introduction to Real Analysis, Wiley Publishers.